Automorphisms and twisted vertex operators

Myhill, Richard Graham

Automorphisms and Twisted Vertex Operators

by

Richard Graham Myhill

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Department of Mathematical Sciences
University of Durham
Durham UK

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This work is dedicated to my parents and
Anne Rebecca Pitt.

The theory of groups and their representations has been recognized for a long time as a source of canonical structures for practically all branches of mathematics and theoretical physics. The theory is remarkable in its internal beauty and every new connection strikes us with the profundity of the symmetry principle underlying so much of our knowledge.

I.B. Frenkel, Beyond Affine Lie Algebras†

† A talk given at the international Congress of Mathematicians. August 3–11, 1986. Berkeley.

ABSTRACT

This work is an examination of various aspects of twisted vertex operator representations of Kac–Moody algebras. It starts with an introduction to Kac–Moody algebras and string theories, including a discussion of the propagation of strings on orbifolds. String interactions in a subclass of such models naturally involve twisted vertex operators.

The centrally extended loop algebra realization of Kac–Moody algebras is used to explain why the inequivalent gradations of basic representations of Kac–Moody algebras $g^{(\tau)}$ associated with g are in one–to–one correspondence with the conjugacy classes of the automorphism group of the root system, aut Φ_g.

The structure of the automorphism groups of the simple Lie algebra root systems are examined. A method of classifying the conjugacy classes of the Weyl groups is explained and then extended to cover the whole automorphism group in cases where there are additional Dynkin diagram symmetries. All possible automorphisms, σ, that have the property that $\det(1-\sigma^r) \neq 0,\ r = 1, \ldots\ldots, n-1$ where n is the order of σ, are determined. Such automorphisms lead to interesting orbifold models in which some of the calculations are simplified.

A thorough exposition of the twisted vertex operator representation is given including a detailed explanation of the zero–mode Hilbert space and the construction of the required cocycle operators. The relation of the vacuum degeneracy to the number of fixed subspace singularities in the orbifold construction is discussed. Explicit examples of twisted vertex operators and their associated cocycles are given.

Finally it is shown how the twisted and an alternative shifted vertex operator representation of the same gradation may be identified. This is used to determine the invariant subalgebras of the gradations along with the vacuum degeneracies and conformal weights of the representations. The results of calculations for inequivalent gradations of the simply laced exceptional algebras are given.

CONTENTS

PREFACE

The work presented in this thesis was carried out between October 1984 and September 1987 in the Department of Mathematical Sciences at the University of Durham, under the supervision of Dr. E.F.Corrigan.

The material in this thesis has not been submitted previously for any degree in this or any other university. No claim of originality is made for the material reviewed in Chapters 1 and 2 or the Appendix. Most of Chapter 3 is drawn from [1] and [2] although Corollary (3.3) and the choice of phases in Lemma (3.6) are due to the author. Chapter 4 is based largely on [3]. However the extension of the investigation to outer automorphisms and the derivation of the matrix representations was undertaken by the author. Chapter 5 is claimed to be original and has been submitted for publication [4]. Chapter 6 is based on [5] but the elucidation of the cocycle construction was carried out with Tim Hollowood [6]. The method of calculation and results of Chapter 7 are original. Some of this work was also done in collaboration with Tim Hollowood and is written up in [7] which has been accepted for publication by the International Journal of Modern Physics A.

I should like to thank Ed Corrigan for his guidance and encouragement throughout the course of this work. I should also like to thank Tim Hollowood for many helpful discussions over the last two years. Thanks are also due to the University of Durham for supporting this work via a Durham University Research Studentship.

0. Introduction.

Twisted vertex operator representations of Kac–Moody algebras arise naturally in a number of areas of mathematics and theoretical physics. In a bosonic form they first arose in physics in an attempt to develop an off–shell formulation of the dual model [8,9,10]. This made use of the fact that fully twisted strings do not conserve momentum. In fact their ancestry could be traced even further back as the Neveu–Schwarz [11,12] sector of the spinning string could be viewed as a twisted relative of the Ramond sector [13]. They reappeared recently, in a more general form, in the study of the propagation of strings on orbifolds [14,15,16]. As such they allow a possible method of dimensional reduction and symmetry breaking in string theories. This is necessary if the currently favoured models [17,18] are to produce a more interesting phenomenology in a physical number of dimensions. Using the particular form of orbifold construction and twisted vertex operator representations that we describe in this work there turns out to be a 112 non-equivalent ways to break E_8.

In mathematics the general construction of twisted (bosonic) vertex operator representations of simply laced Kac–Moody algebras has been elucidated [5,19]. Such constructions give non-integer graded representations of the Kac–Moody algebras. Chronologically they actually arose before the untwisted representations in a representation of the twisted algebra $A_1^{(1)}$ [20]. The untwisted construction was given in [21,22]. In the last few years a combination of $\mathbb{Z}_2$ twisted and untwisted vertex operators has been used to construct the so called 'moonshine' representation of the monster group, F_1; the largest sporadic finite simple group [23,24]. Thus again there seems to be a mysterious connection between strings and an area of pure mathematics. Strings and finite simple groups already have interesting mutual connections via modular functions.

The idea of this thesis is to try to draw together some of the ideas associated with the mathematics underlying these constructions. As well as reviewing various topics and constructions we determine some of the resulting invariant algebras and distinguish an interesting subset of representations.

Chapter 1 consists of an introduction to Kac–Moody and Lie algebras. It establishes ideas and terminology that are needed later in the work. It includes a detailed look at the possible choices for the structure constants. In particular it is shown that they can be chosen to be integers (Chevalley basis) and for the case of simply laced algebras that a 2–cocycle with a few additional conditions provides a suitable set. Finally the restrictions put on the generators and structure constants by hermiticity conditions are examined. A number of possible conventions are elucidated

In the next chapter we give a brief introduction to string theories. This includes an elementary review of both the bosonic and the heterotic strings. We then relax some of the *physical* constraints on string theories and look at the general propagation of strings on orbifolds. These are spaces obtained by quotienting out a manifold by the action of a discrete group. In particular we look at quotienting $\mathbf{R}^n$ by a space group consisting of the semi-direct product of the group of translations by roots in the root lattice of a simply laced Lie algebra and some subgroup of the automorphism group of the root system, aut Φ_g. It is seen that the resulting string theory has twisted sectors corresponding to strings that only close up to an element of the space group. The interaction of such strings involves the construction of twisted vertex operators which motivates our interest in such objects. In addition the original symmetry group that the string states at a given mass level form representations of is broken to some subgroup G_0. It is therefore gives a possible method for symmetry breaking in string theories as well as a way of dimensional reduction.

In Chapter 3 we show how to realize an arbitrary integer gradation of an infinite dimensional Kac–Moody algebra in terms of its underlying finite Lie algebra. This involves a preliminary study of Lie algebra automorphisms and in particular those of finite order. We give a classification of all finite order automorphisms and a method of determining the resulting invariant subalgebras both of which are due to Kac [25,1]. We establish some 'machinery' and phase conventions that are used in later calculations. It is also shown that for each conjugacy class of aut Φ_g there is an inequivalent gradation of a Kac–Moody algebra $g^{(\tau)}$ associated with g. Finally we realize the Kac–Moody algebras in terms of centrally extended loop algebras and their subalgebras.

Chapter 4 consists of a study of the groups of automorphisms of Lie algebra root systems. After the general structure of such groups is given we go on to give an exposition of a classification of the conjugacy classes of the Weyl groups in terms of some Dynkin diagram like graphs due to Carter [3]. These can be used to construct explicit elements in each conjugacy class. We then extend our investigation to conjugacy classes of the full automorphism groups which are not given in [3]. We elucidate all the cases of which that of aut Φ_{D_4} is the most interesting and hence dealt with in some detail. The chapter ends with a few observations on the matrix representations of aut Φ_g for the classical Lie algebras.

The twisted vertex operators corresponding to a given automorphism σ are greatly simplified if σ leaves no point but the origin fixed. A subset of these automorphisms are those for which an arbitrary power of σ leaves nothing but the origin fixed (or is equal to the identity element), *i.e.* $\det(1\text{-}\sigma^r) \neq 0$ $r = 1, ..., n-1$, where n is the order of σ. We call such automorphisms no fixed point automorphisms or NFPAs. It is shown in Chapter 2 that an interesting subclass of orbifold models is given when we take the point group to be generated by a NFPA as they lead to string theories in which the momentum is eliminated in all the twisted sectors. In Chapter 5 we classify all the NFPA of Lie algebra root systems [4]. We conclude the chapter by looking at some nice properties of the third order NFPAs.

Chapter 6 fleshes out a twisted vertex operator construction given in [5]. We start by reviewing the ordinary vertex operator representation of Kac–Moody algebras before proceeding to the more complicated twisted vertex operator representations. We sketch in some detail the calculations that are necessary to show that the moments of such operators truly give a representation of a Kac–Moody algebra. To do this we give a detailed account of the zero–mode Hilbert space and the cocycle operators on this space that are required for the construction to work. We also discuss how the vacuum degeneracy is related to the number of fixed points in the orbifold construction. We include some explicit examples of twisted vertex operator constructions and their associated cocycle operators.

In Chapter 7 we introduce an alternative shifted vertex operator construction of the graded representations of simply–laced Kac–Moody algebras corresponding to inner root system automorphisms. We explain how it is possible to calculate the invariant subalgebra and how to determine the vacuum degeneracies of the corresponding representations. As an example of the method we give results for all the *different* gradations, corresponding to conjugacy classes of the associated roots system automorphism group aut Φ_g, of the simply laced exceptional algebras E_6, E_7 and E_8 . These extend the results given in [7].

The work finishes with a summary of what has been achieved and a few final comments in Chapter 8. These include the possibility of the original symmetry of the string being restored by the combination of various sectors of the theory, the use of twisted vertex operators in mathematics and some generalisations of both the twisted vertex operator and orbifold constuctions.

1. An Introduction to Kac-Moody and Lie Algebras.

The first part of this chapter consists of an introduction to Kac–Moody algebras. It introduces some definitions and terminology that are used later, see for example [1,2]. We then go on to look at the special case of Lie algebras particularly in the Cartan–Weyl and Chevalley bases. We also examine the restrictions put on the structure constants by the Jacobi identity and hermiticity considerations.

1.1 A BRIEF INTRODUCTION TO KAC-MOODY ALGEBRAS.

Let $A = (a_{ij})$ be an n×n complex matrix with;

$$\begin{array}{llll} (1) & a_{ii} = 2 & \text{for } i = 1, \dots, n \\ (2) & a_{ij} \leq 0 & \text{for } i \neq j \\ (3) & a_{ij} = 0 \Rightarrow a_{ji} = 0 & \end{array}$$

Such a matrix is known as a **generalized Cartan** (or **GCM**).

The complex **Kac-Moody algebra** g(A) associated with A is then the algebra generated from the complex vector space spanned by $\{h_i, e_i^+, e_i^- \mid i = 1, \dots, n\}$ by use of the commutation relations,

$$[h_i, h_j] = 0 \tag{1.1}$$

$$[h_i, e_j^\pm] = \pm a_{ij} e_j^\pm \tag{1.2}$$

$$[e_i^+, e_j^-] = \delta_{ij} h_j \tag{1.3}$$

$$(\mathbf{Ad}e_i^\pm)^{1-a_{ij}} e_j^\pm = 0 \tag{1.4}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in g(A) \quad \textbf{Jacobi Identity} \tag{1.5}$$

where $\mathbf{Ad}e_i^\pm$ are the linear maps

$$\mathbf{Ad}e_i^\pm : g(A) \to g(A) \quad \text{such that}$$
$$\mathbf{Ad}e_i^\pm(x) = [e_i^\pm, x]$$

The rank of the matrix A divides the resulting algebras, g(A), into three distinct types:

(1) det A $\neq 0$, *i.e.* rank A $=$ n: g(A) is finite dimensional. In this case A is invertible and g(A) is a finite dimensional simple Lie algebra. All the general terminology we introduce reduces to the normal definitions for Lie algebras when det A $\neq 0$.

(2) det A $= 0$, rank A $= n-1$: g(A) is infinite dimensional. Here g(A) is known as an **affine Kac-Moody algebra.**

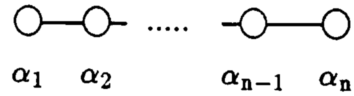
(3) det A $= 0$, rank A $< n-1$: g(A) is infinite dimensional. An example of such an algebra is the Lorentzian algebra E_{10} [26]. We shall not be interested any further in this case.

All the information in a GCM can be described by a diagram called a **Dynkin diagram.** This is a diagram with n nodes, with each pair of nodes (i, j) being connected by $\max(|a_{ij}|, |a_{ji}|)$ lines with an arrow pointing from i to j if $|a_{ij}| < |a_{ji}|$. We shall denote the Dynkin diagram of g by D(g).

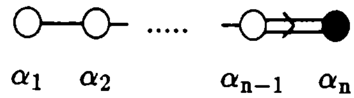
It is not too difficult to classify all the possible diagrams/algebras. They are listed in Table 1.1. The labeling is of the form $g^{(\tau)}$ where g is a finite simple Lie algebra and τ is a number called the **twist** associated with the algebra. The diagrams with $\tau = 0$ are those of the usual finite simple Lie algebras and in this case the superscript is omitted. The $\tau = 1$ diagrams, called the **extended Dynkin diagrams** of g, are obtained from the $\tau = 0$ diagrams by appending minus the highest root of the Lie algebra g, $-\Theta$, to its Dynkin diagram. The corresponding algebras are quite often just written as $\hat{g}$. To obtain the $\tau = 2, 3$, which are called **twisted algebras** (as opposed to the $\tau = 0, 1$ **untwisted algebras**) one has to make use of the symmetry of the Dynkin diagram of g. We shall explain how to do this in Chapter 2. Notice that removing one spot from any infinite dimensional algebra's diagram leaves the diagram of a, not necessarily simple, finite algebra.

Table 1.1 : *Dynkin diagrams for the affine and finite Kac-Moody algebras.*

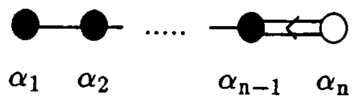
$\underline{\tau = 0:}$ A_n :



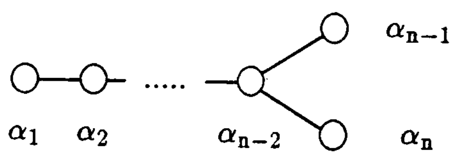
B_n :



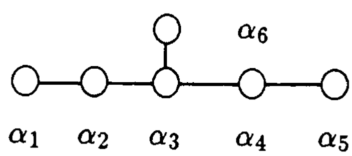
C_n :



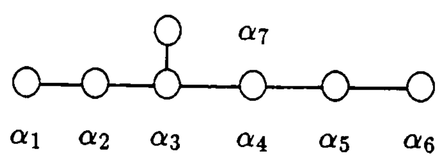
D_n :



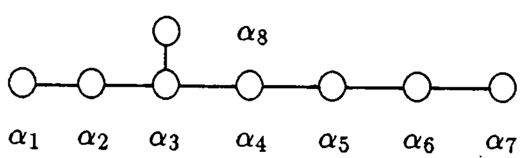
E_6 :



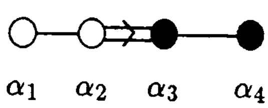
E_7 :



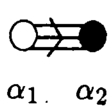
E_8 :



F_4 :



G_2 :



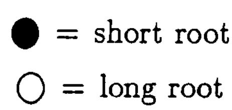
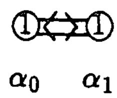


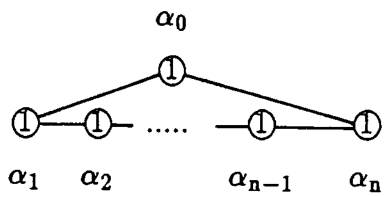
Table 1.1 : *Continued.*

$\tau = 1$:

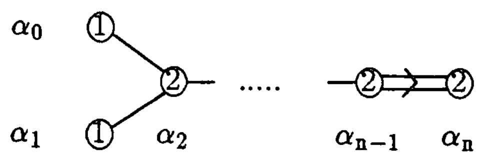
$A_1^{(1)}$:



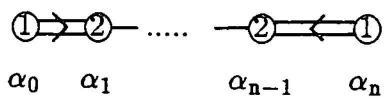
$A_n^{(1)}$ $(n \geq 2)$:



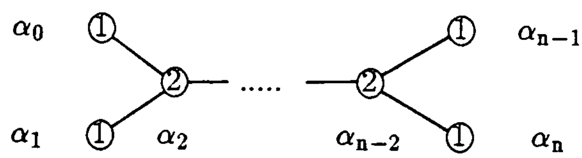
$B_n^{(1)}$ $(n \geq 3)$:



$C_n^{(1)}$ $(n \geq 2)$:



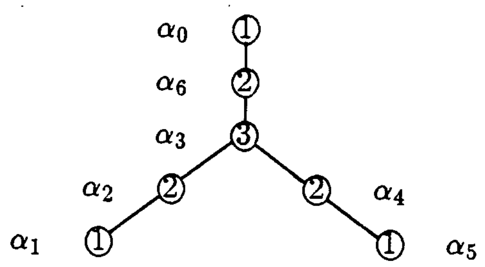
$D_n^{(1)}$ $(n \geq 4)$:



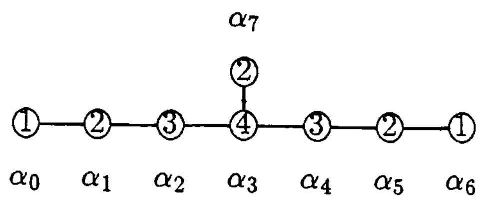
(n) = Kac label.

Table 1.1 : *Continued.*

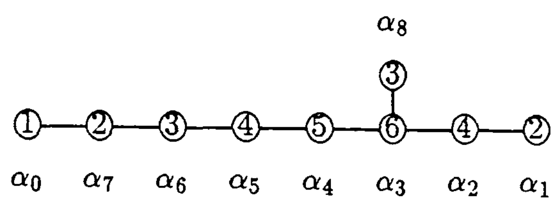
$\tau = 1$ (cont.) : $E_6^{(1)}$:



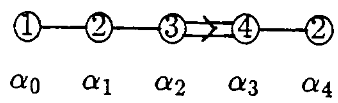
$E_7^{(1)}$:



$E_8^{(1)}$:



$F_4^{(1)}$:



$G_2^{(1)}$:

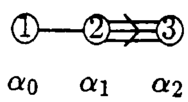
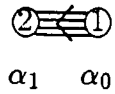
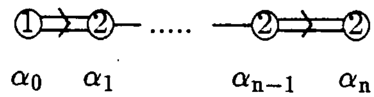


Table 1.1 : *Continued.*

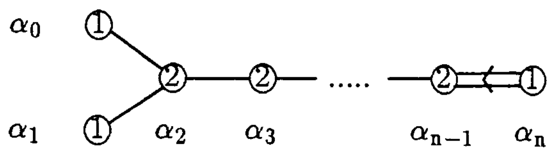
$\underline{\tau = 2:}$ $A_2^{(2)}$:



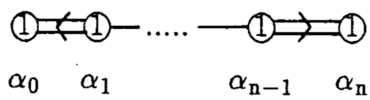
$A_{2n}^{(2)}$ $(n \geq 2)$:



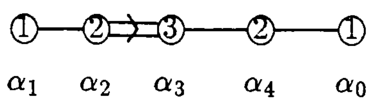
$A_{2n-1}^{(2)}$ $(n \geq 3)$:



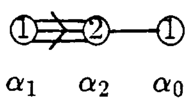
$D_{n+1}^{(2)}$ $(n \geq 2)$:



$E_6^{(2)}$:



$\underline{\tau = 3:}$ $D_4^{(3)}$:



Now for each $g(A)$ we have a **dual algebra**, $g^D(A)$, obtained by interchanging the long and the short roots, that is replacing A by A^T, thus $g^D(A) = g(A^T)$. The Dynkin diagram of $g^D(A)$ is easily found from that of $g(A)$ by reversing all the arrows.

The vector space generated by the h_i's, $H \equiv \{c_i h_i \mid c_i \in \mathbb{C}\}$, is a maximal commuting subalgebra of $g(A)$ known as the **Cartan subalgebra** (or **CSA**). Its dual space, H^*, is called the **root space**. We use A to identify these two spaces. To each $h_i \in H$ we associate an $\alpha_i \in H^*$ such that

$$\alpha_i(h_j) = a_{ji}.$$

The vector α_i is known as a **simple root**, in particular it is the simple root associated with the generator e_i^+. We denote the set of simple roots by Δ.

The lattice formed by the integral span of the simple roots, $\Lambda_R \equiv \{n_i\alpha_i \mid n_i \in \mathbb{Z}\} \subset H^*$ is called the **root lattice**. We define a positive subset of the root lattice as follows,

$$\Lambda_R^+ \equiv \{n_i\alpha_i \in \Lambda_R \mid n_i \in \mathbb{Z} \text{ and first non-zero } n_i \text{ is positive}\}.$$

We can thus introduce an ordering, $>$, on Λ_R by defining

$$\alpha > \beta \quad \text{iff} \quad \alpha - \beta \in \Lambda_R^+.$$

If $\alpha = \sum_{i=1}^{n} n_i\alpha_i \in \Lambda_R$ then the **height of** α is defined as

$$\mathrm{ht}(\alpha) \equiv \sum_{i=1}^{n} n_i.$$

Notice that $\alpha > \beta$ implies that $\mathrm{ht}(\alpha) > \mathrm{ht}(\beta)$ but the converse is *not* true.

The elements $\{e_i^{\pm}\}$ are known as **simple step operators** or **Chevalley generators**, the $\{e_i^+\}$ being called **simple raising operators** and the $\{e_i^-\}$ **simple lowering operators**.

Let $n_\pm$ be the subalgebras of $g(A)$ generated by $\{e_i^+\}$ and $\{e_i^-\}$ respectively. $g(A)$ has the triangular decomposition,

$$g(A) = n_- \oplus H \oplus n_+.$$

$n_\pm$ can be further decomposed into eigenspaces of H. Let

$$g_\alpha \equiv \{x \in g(A) \mid [h, x] = \alpha(h)x \ \forall h \in H\},$$

then we have the following **root space decomposition** of $g(A)$,

$$g(A) = \left(\bigoplus_{\alpha \in \Lambda_R^+ - \{0\}} g_{-\alpha}\right) \oplus H \oplus \left(\bigoplus_{\alpha \in \Lambda_R^+ - \{0\}} g_\alpha\right),$$

or

$$g(A) = \bigoplus_{\alpha \in \Lambda_R} g_\alpha,$$

as $H = g_0$. α is known as the **root** associated with x.

The sublattice of Λ_R such that $g_\alpha \neq \emptyset$ is called the **root system**, Φ_g, of $g(A)$ *i.e.*

$$\Phi_g \equiv \{\alpha \in \Lambda_R \mid g_\alpha \neq \emptyset\}.$$

We denote the positive and negative roots by

$$\Phi_g^\pm \equiv \pm\Lambda_R^+ \cap \Phi_g.$$

Note that $\Phi_g = \Phi_g^- \cup \Phi_g^+$ and $\Phi_g^- \cap \Phi_g^+ = \emptyset$.

We can see that:

1. If $\det A \neq 0$ then the root system is non–degenerate *i.e.* $\dim g_\alpha = 1 \; \forall \; \alpha \in \Phi_g$. In addition $\mid \Phi_g \mid$ is finite. In this case the element of Φ_g with the greatest height is known as the **highest root**.

2. However if rank $A = n-1$ then the rows and columns of A are linearly dependent. Hence there exist a unique set of integers, k_i, called **Kac labels** and k_i^v, called **dual Kac labels** such that

$$\sum_{j=0}^{n} a_{ij}k_j = \sum_{j=0}^{n} k_j^v a_{ji} = 0 \qquad i = 0, \ldots\ldots, n$$

and $\min(k_i) = \min(k_i^v) = 1$.

(a) The **Coxeter and dual Coxeter numbers**, h_A and h_A^v, of A (or g(A)) are defined to be the sum of these labels,

$$h_A \equiv \sum_{i=0}^{n} k_i \qquad h_A^v \equiv \sum_{i=0}^{n} k_i^v.$$

(b) The Kac labels are given in Table 1.1. The dual Kac labels of g(A) are just the Kac labels of the dual algebra $g(A^T)$. Also a given k_i is equal to half the sum of the adjacent k_js appropriately weighted by the number of lines joining i to j if there is an arrow pointing from j to i.

Because of this linear dependence an element of the CSA, called the **central element** or **central charge**, defined by

$$C \equiv \sum_{i=0}^{n} k_i^v h_i$$

forms a one–dimensional centre that commutes with the whole algebra g(A). There is only one such element as rank A $= n-1$. Consequently $\alpha(C) = 0 \ \forall \alpha \in H^*$. Similarly there is a **null root,**

$$\delta \equiv \tau \sum_{i=0}^{n} k_i \alpha_i,$$

such that $\delta(h) = 0 \ \forall h \in H$. Thus all the roots of g(A) are *infinitely* degenerate. In particular $g_\alpha = g_{\alpha+n\delta}$.

To remove this degeneracy we can extend g(A) to $g(A)\otimes\mathbb{C}d$ by adding a **derivation** d to its CSA, where we choose

$$[d, e_i^{\pm}] = \pm\Delta_i e_i^{\pm}, \tag{1.6}$$

$$[d, h_i] = [d, d] = 0, \tag{1.7}$$

where $\Delta_i \in \mathbb{Z}$, with at least one $\Delta_i \neq 0$ and there is no summation implied in (1.6). (1.6) is equivalent to choosing $\alpha_i(d) = \Delta_i$. Let $[d, x] = \lambda x$, $[d, y] = \mu y$ then

$$\begin{aligned}[d, [x, y]] &= [[d, x], y] + [x, [d, y]], \\ &= (\mu + \lambda)[x, y].\end{aligned}$$

Thus there is a natural **gradation** of g associated with the derivation d,

$$g = \bigoplus_{n \in Z} g_n \tag{1.8}$$

where $g_n \equiv \{x \in g \mid [d, x] = nx\}$. What is more, $\dim g_n$ is finite for all n. We thus have a finite dimensional Lie subalgebra g_0 of g(A), sometimes called the **horizontal**

algebra, which commutes with d,

$$[\mathrm{d}, \mathrm{g}_0] = 0.$$

All the other levels must form representations of g_0 as,

$$[\mathrm{g}_0, \mathrm{g}_\mathrm{n}] \subset \mathrm{g}_\mathrm{n}.$$

We call (1.6) and (1.7) a **gradation of g(A) of type** Δ. A gradation of $\mathrm{g}^{(\tau)}$ can alternatively be thought of as an eigenspace decomposition of $\mathrm{g}^{(\tau)}$ under some automorphism. It is this approach that we follow in the next chapter.

There are two gradations which are worth a special mention.

1. If $\mathrm{k}_j = 1$ and we choose $\Delta_i = \delta_{ij}$ then we get what is known as a **homogeneous gradation**. The untwisted algebras have a unique homogeneous gradation upto isomorphism, whilst the twisted algebras $\mathrm{A}^{(2)}_{2\mathrm{n}-1}$, $\mathrm{E}^{(2)}_6$ and $\mathrm{D}^{(3)}_4$ have two and $\mathrm{D}^{(2)}_{\mathrm{n}+1}$ has $\left[\frac{\mathrm{n}}{2}\right]$. g_0 is the finite Lie algebra whose Dynkin diagram is obtained by removing the j^{th} node from the Dynkin diagram of the Kac–Moody algebra $\mathrm{g}^{(\tau)}$.

$$\mathrm{g}_n \cong \mathrm{g}_0 \qquad \forall\, \mathrm{n} \in \mathbb{Z} - \{0\}.$$

2. If we take $\Delta_i = 1\ \forall i$ then we get the **principal gradation** with $\mathrm{g}_0 = [\mathrm{U}(1)]^\mathrm{n}$.

Let us introduce the **fundamental weights** $\Lambda_i \in \mathrm{H}^*$ of $\mathrm{g}^{(\tau)}$. They are defined by,

$$\Lambda_i(\mathrm{h}_j) = \delta_{ij}.$$

Λ_0, called the **highest weight of the basic representation**, does not belong to the space spanned by the α_i $i = 0, ..., \mathrm{n}$ but is defined to be another null vector that is orthogonal to all the α_i with $i = 1, ..., \mathrm{n}$ and to be dual to the light–like vector δ *i.e.* $(\Lambda_0,\ \delta) = \tau$. The vector space H^* spanned by Λ_0, δ, and α_i $i = 1, ..., \mathrm{n}$ is then an (n+1)–dimensional Lorentzian space where Λ_0 and δ are dual light–like vectors.

Let $\mathrm{R}(\mathrm{g}^{(\tau)})$ be a representation of the Kac–Moody algebra $\mathrm{g}^{(\tau)}$. The gradation on $\mathrm{g}^{(\tau)}$ gives a gradation on the the representation space. A representation in which this gradation is bounded *below* is called a **highest weight representation.** Such representations are interesting because in physical applications the derivation is usually considered to be either an *energy* or a *scale* operator. They can be built up from a **highest weight state,** $|\Lambda\rangle$ where $\Lambda = \sum_{i=0}^{n} \delta_i \Lambda_i$. The components δ_i are known as the **Dynkin weights** of Λ. A weight is said to be **dominant** if $\delta_i \geq 0, \quad i = 1, ..., n$. In this representation we have,

$$\begin{aligned} h_i \mid \Lambda\rangle &= \delta_i \mid \Lambda\rangle, \\ e_i^+ \mid \Lambda\rangle &= 0. \end{aligned}$$

The representation space consists of the complex linear span of states of the form

$$e_{i_1}^- e_{i_2}^- .... e_{i_r}^- \mid \Lambda\rangle$$

and is known as a **Verma module,** $V(\Lambda)$. It is characterised by the highest root of its horizontal subalgebra and the eigenvalue of the central term on the vacuum, called the **level** of the representation,

$$\begin{aligned} C \mid \Lambda\rangle &= K \mid \Lambda\rangle, \\ &= \sum_{i=0}^{n} \delta_i k_i^v \mid \Lambda\rangle. \end{aligned}$$

In general a highest weight representation is reducible. Just as for the case of finite Lie algebras the resulting representation is **unitary** if and only if the Dynkin weights are non–negative integers [1]. In this case Λ is known as an **integral weight** and $K \geq 0$ is an integer with $K = 0$ corresponding to the trivial representation. Level one representations are also known as **basic representations.** They are basic in the sense that all other unitary highest weight representations can be built up from them by taking direct products of such representations and reducing them into invariant subspaces. In addition we have vertex operator representations of the basic representations of simply laced Kac–Moody algebras [21,22]. Notice that the level depends on the *dual* Kac labels, so for example $E_6^{(2)}$ only has one level one representation. If a given Kac–Moody algebra has two or more dual Kac labels which are all equal to one and can be transformed into each other by a symmetry of the Dynkin diagram then they lead to isomorphic basic representations.

Given a particular representation there are many possible gradations of it corresponding to the choice of derivation. This is in contrast to the level of the representation which is intrinsically defined. In this work we examine the twisted vertex operator representations corresponding to different gradations of basic representations of simply laced Kac–Moody algebras. Up to conjugacy of the Heisenberg subalgebras, the inequivalent gradations of basic representations of Kac–Moody algebras associated with the underlying Lie algebra g are in one–to–one correspondence with the conjugacy classes of the automorphism group of the root system of g, aut Φ_{g} [16]. The **vacuum states**, that is the states with zero gradation, will vary with our choice of derivation. They must form a representation of the horizontal subalgebra. This representation has to be irreducible if the Kac–Moody algebra representation is to be irreducible.

1.2 Lie algabras: Cartan–Weyl basis.

Given a CSA H of a Lie algebra g, a basis $\{\mathrm{h}_i, \mathrm{E}_\alpha \mid \mathrm{h}_i \in \mathrm{H},\ i = 1, ..., \mathrm{rank}\,\mathrm{g},\ \alpha \in \Phi_{\mathrm{g}}\}$ can be chosen for g such that the commutation relations are:

$$
\begin{aligned}
[\mathrm{h}_i, \mathrm{h}_j] &= 0 && (1.9)\\
[\alpha.\mathrm{H}, \mathrm{E}_\beta] &= \alpha.\beta \mathrm{E}_\beta && (1.10)\\
[\mathrm{E}_\alpha, \mathrm{E}_\beta] &= \begin{cases} \mathrm{N}_{\alpha,\beta}\mathrm{E}_{\alpha+\beta} & \alpha+\beta \in \Phi_{\mathrm{g}} \\ \mathrm{B}_\alpha \alpha.\mathrm{H} & \alpha+\beta = 0 \\ 0 & \alpha+\beta \notin \Phi_{\mathrm{g}} \end{cases} && (1.11)
\end{aligned}
$$

where;

$$\mathrm{N}_{\alpha,\beta} = -\mathrm{N}_{\beta,\alpha}. \tag{1.12}$$

$$\mathrm{N}_{\alpha,\beta}\mathrm{B}_\gamma, = \mathrm{N}_{\beta,\gamma}\mathrm{B}_\alpha = \mathrm{N}_{\gamma,\alpha}\mathrm{B}_\beta \text{ if } \alpha,\beta,\gamma \in \Phi_{\mathrm{g}} \text{ and } \alpha+\beta+\gamma = 0. \tag{1.13}$$

$$\mathrm{N}_{\alpha,\beta}\mathrm{N}_{\gamma,\delta}\mathrm{B}_{\alpha+\beta} + \mathrm{N}_{\beta,\gamma}\mathrm{N}_{\alpha,\delta}\mathrm{B}_{\beta+\gamma} + \mathrm{N}_{\gamma,\alpha}\mathrm{N}_{\beta,\delta}\mathrm{B}_{\alpha+\gamma} = 0 \tag{1.14}$$

if $\alpha,\beta,\gamma,\delta \in \Phi_{\mathrm{g}},\ \alpha+\beta+\gamma+\delta = 0$ and no sum of two of them is zero.

$$\mathrm{N}_{\alpha,\beta}\mathrm{N}_{-\alpha,-\beta} = -\frac{\alpha^2}{2}\frac{\mathrm{B}_\alpha \mathrm{B}_\beta}{\mathrm{B}_{\alpha+\beta}}\mathrm{q}(\alpha,\beta)\{\mathrm{p}(\alpha,\beta)+1\}. \tag{1.15}$$

and

(1) $\alpha.\mathrm{H} = n_i\mathrm{h}_i$ if $\alpha = \sum_{i=1}^{\mathrm{rank}\,\mathrm{g}} n_i\alpha_i$ is the decomposition of α in terms of simple roots.

(2) $\mathrm{B}_\alpha \equiv \mathrm{Tr}\{\mathbf{ad}(\mathrm{E}_\alpha)\,\mathbf{ad}(\mathrm{E}_{-\alpha})\}$, $\alpha \in \Phi_{\mathrm{g}}$, where $\mathbf{ad}(\mathrm{x})$ is the *adjoint representation* of $\mathrm{x} \in \mathrm{g}$ and Tr denotes a trace. In general $\mathrm{B}{:}\mathrm{g}\times\mathrm{g} \to \mathbb{C}$ such that $\mathrm{B(a,b)} \equiv \mathrm{Tr}\{\mathbf{ad}(\mathrm{a})\,\mathbf{ad}(\mathrm{b})\}$ is a *symmetric bilinear form* on g called the **Killing form**. Because $\mathrm{B}(\mathrm{E}_\alpha, \mathrm{E}_\alpha) = 0$ the Killing form does not give a normalization for the E_α, so we can choose $\mathrm{B}_\alpha = \mathrm{B}(\mathrm{E}_\alpha, \mathrm{E}_{-\alpha})$ arbitrarily for each $\alpha \in \Phi_{\mathrm{g}}$. In the Cartan-Weyl basis we can choose the h_i such that,

$$\begin{aligned}
\mathrm{B}(\mathrm{E}_\alpha, \mathrm{E}_\beta) &= \mathrm{B}_\alpha \delta_{\alpha+\beta,0}, \\
\mathrm{B}(\mathrm{E}_\alpha, \mathrm{h}_i) &= 0, \\
\mathrm{B}(\mathrm{h}_i, \mathrm{h}_j) &= \delta_{ij}.
\end{aligned}$$

(3) The inner product on the root space is obtained from the Killing form restricted to H, which is an inner product on H, by duality. $\alpha.\beta \equiv \mathrm{B}(\mathrm{h}_\alpha, \mathrm{h}_\beta)$ where $\mathrm{h}_\alpha, \mathrm{h}_\beta \in \mathrm{H}$ are the elements dual to $\alpha, \beta \in \mathrm{H}^*$ respectively.

(4) $\mathrm{p}(\alpha,\beta), \mathrm{q}(\alpha,\beta) \in \mathbb{Z}^+ + \{0\}$ are defined by the **α-chain of roots through β**, the sequence of roots,

$$\beta - \mathrm{p}(\alpha,\beta)\alpha, \ldots\ldots, \beta, \ldots\ldots, \beta + \mathrm{q}(\alpha,\beta)\alpha$$

such that

$$\beta - (\mathrm{p}(\alpha,\beta)+1)\alpha,\ \beta + (\mathrm{q}(\alpha,\beta)+1)\alpha \notin \Phi_{\mathrm{g}}.$$

Proof : See for eg [27], in particular pp 829-831.

For a given choice of CSA there is a huge amount of freedom left in choosing a Cartan-Weyl basis. Given a particular basis we can:

(1) : Make a change of basis in the CSA. However if we wish it to remain orthonormal with respect to the Killing form we must choose an orthogonal transformation.

(2) : Map

$$\begin{aligned}
\mathrm{E}_\alpha &\mapsto \eta(\alpha)\mathrm{E}_{\sigma(\alpha)}, \\
\mathrm{N}_{\alpha,\beta} &\mapsto \frac{\eta(\alpha)\eta(\beta)}{\eta(\alpha+\beta)}\mathrm{N}_{\sigma(\alpha),\sigma(\beta)}, \\
\mathrm{B}_\alpha &\mapsto \eta(\alpha)\eta(-\alpha)\mathrm{B}_\sigma(\alpha),
\end{aligned}$$

where $\sigma \in \text{aut}\,\Phi_g$ is an automorphism of the root system and $\eta : \Phi_g \to \mathbb{C} - \{0\}$. If in addition we want to keep the structure constants fixed, *i.e.* $N_{\alpha,\beta} \mapsto N_{\alpha,\beta}$, then we must have

$$\eta(\alpha)\eta(\beta) = \frac{N_{\alpha,\beta}}{N_{\sigma(\alpha),\sigma(\beta)}}\eta(\alpha+\beta).$$

So the $\eta(\alpha)$ must form a *projective representation* of Φ_g with the factor set $\Big\{H(\alpha,\beta) \equiv \frac{N_{\alpha,\beta}}{N_{\sigma(\alpha),\sigma(\beta)}}\Big\}$.

(3) : Make an *independent* choice of each of the B_α ($\in \mathbb{C}$). This choice fixes the moduli of the step operators but leaves their phases undetermined. So we still have the freedom to map

$$\begin{aligned} E_\alpha &\mapsto \mu(\alpha)E_\alpha, \\ N_{\alpha,\beta} &\mapsto \frac{\mu(\alpha)\mu(\beta)}{\mu(\alpha+\beta)}N_{\alpha,\beta}, \end{aligned}$$

where $\mu : \Phi_g \to S^1$, $S^1 = \{x \in \mathbb{C} \mid |\, x \,| = 1\}$ and $\mu(\alpha)\mu(-\alpha) = 1$. If in addition the structure constants are fixed then μ must be a homomorphism,

$$\mu(\alpha)\mu(\alpha) = \mu(\alpha+\beta).$$

Chevalley basis :

By case by case examination of all the algebras (see for eg p 54 of [28]) we have,

$$\frac{(\alpha+\beta)^2}{\beta^2}q(\alpha,\beta) = p(\alpha,\beta)+1$$

Thus by choosing,

$$\begin{aligned} B_\alpha &= \frac{2}{\alpha^2}, \\ N_{-\alpha,-\beta} &= -N_{\alpha,\beta}, \end{aligned} \qquad \text{(I)}$$

or

$$\begin{aligned} B_\alpha &= -\frac{2}{\alpha^2}, \\ N_{-\alpha,-\beta} &= N_{\alpha,\beta}, \end{aligned} \qquad \text{(II)}$$

We have from (1.15),

$$N_{\alpha,\beta} = \pm(p(\alpha,\beta)+1).$$

Note : We shall show that such a choice of $N_{\alpha,\beta}$ and B_α is consistent in the next subsection, 1.2.1.

In such a special choice of Cartan-Weyl basis, called a **Chevalley basis**, all the structure constants are integers. Case **(I)** is the most usual choice because it means we have the usual hermiticity conditions on the generators (see 1.2.2.).

Now under a Weyl reflection in the root α, the α-chain through β is reversed so that,

$$-2\alpha.(\beta - p(\alpha,\beta)\alpha) = 2\alpha.(\beta + q(\alpha,\beta)\alpha)$$

and thus,

$$p(\alpha,\beta) - q(\alpha,\beta) = \frac{2\alpha.\beta}{\alpha^2}.$$

However if $\alpha+\beta \in \Phi_g$ then $q(\alpha,\beta) \geq 1$ so that

$$1 \leq | N_{\alpha,\beta} | \leq \left| \frac{2\alpha.\beta}{\alpha^2} \right|.$$

Table 1.2 : *Possible values of the Chevalley structure constants.*

A_n, D_n, E_6, E_7, E_8.	± 1.
B_n, C_n, F_4.	± 1, ± 2.
G_2.	± 1, ± 2, ± 3.

It is possible to rescale the structure constants of B_n and F_4 so that they are both just drawn from the set $\{\pm 1\}$. To do this choose $B_\alpha = \pm 1 \ \forall \alpha \in \Phi_g$, in place of $B_\alpha = \pm\frac{2}{\alpha^2}$.

1.2.1 Restrictions on the structure constants.

Theorem (1.1) : For an arbitrary choice of B_α, $\alpha \in \Phi_g^+$, it is always possible to choose the basis elements E_α so that

$$\begin{aligned} &\text{either} \qquad \text{(i) } N_{\alpha,\beta} = N_{-\alpha,-\beta} \qquad \forall\, \alpha,\beta \in \Phi_g, \\ &\text{or} \qquad \text{(ii) } N_{\alpha,\beta} = -N_{-\alpha,-\beta} \quad \forall\, \alpha,\beta \in \Phi_g. \end{aligned}$$

Proof : Let $\sigma : \Phi_g \to \Phi_g$ such that $\sigma(\alpha) = -\alpha \;\; \forall\, \alpha \in \Phi_g$. We can always extend this to an automorphism of the whole algebra (see Chapter 2), $\Sigma : g \to g$, with

$$\begin{aligned} H &\mapsto -H, \\ E_\alpha &\mapsto \psi_\alpha E_{-\alpha}. \end{aligned}$$

Let $\lambda = \pm 1$ and make the change of basis

$$\hat{E}_\alpha \equiv \sqrt{\lambda\psi_{-\alpha}}E_\alpha\ , \qquad \hat{E}_{-\alpha} \equiv \frac{1}{\sqrt{\lambda\psi_{-\alpha}}}E_{-\alpha}\ ,$$

$$N_{\alpha,\beta} \mapsto \frac{\sqrt{\lambda\psi_{-\alpha}}\sqrt{\lambda\psi_{-\beta}}}{\sqrt{\lambda\psi_{-(\alpha+\beta)}}}N_{\alpha,\beta}\ .$$

This leaves the choice of B_α unaltered as

$$B(\hat{E}_\alpha, \hat{E}_{-\alpha}) = B(E_\alpha, E_{-\alpha}) = B_\alpha.$$

Also under σ we have,

$$\begin{aligned} \hat{E}_\alpha &\mapsto \sqrt{\lambda\psi_{-\alpha}}\psi_\alpha\sqrt{\lambda\psi_{-\alpha}}\hat{E}_{-\alpha} = \lambda\hat{E}_{-\alpha}, \\ \hat{E}_{-\alpha} &\mapsto \frac{\psi_{-\alpha}}{\sqrt{\lambda\psi_{-\alpha}}}\frac{1}{\sqrt{\lambda\psi_{-\alpha}}} = \lambda\hat{E}_\alpha. \end{aligned}$$

where we have used the fact that $\psi_{-\alpha}\psi_\alpha = 1$. Now Σ is an automorphism of g and therefore must still preserve the commutation relations even after the change of basis. In particular (1.11) gives

$$\lambda^2 N_{-\alpha,-\beta} = \lambda N_{\alpha,\beta} \;\; i.e. \;\; N_{\alpha,\beta} = \lambda N_{-\alpha,-\beta}.$$

Given a set of B_α, which fixes the magnitudes of the structure constants, and having made a choice of $N_{\alpha,\beta} = \pm N_{-\alpha,-\beta}\ \forall \alpha,\beta \in \Phi_g$. We still have some freedom in the phases of the $N_{\alpha,\beta}$ corresponding to the freedom in phases of the step operators. That is we can multiply all the $N_{\alpha,\beta}$ by a function $f(\alpha,\beta) \equiv \frac{u(\alpha)u(\beta)}{u(\alpha+\beta)}$, where $u : \Phi_g \to S^1$ is an arbitrary function with $u(\alpha) = u(-\alpha)$.

We shall examine the case of a Chevalley basis a little more closely. Every positive non-simple root, *i.e.* $\alpha \in \Phi_g^+ - \Delta$, can be expressed as the sum of two positive roots, perhaps in many different ways. If we choose a specific decomposition, $\alpha = \beta + \gamma$, for each such root then we are free to prescribe the signs of all the corresponding $N_{\beta,\gamma}$. Once we have done this the signs of all the other structure constants are determined (p 54 [29]).

We call such a choice of signs a **normalisation** of the $N_{\alpha,\beta}$. A given normalisation of the $N_{\alpha,\beta}$ is only unique up to multiplication by a function $f(\alpha,\beta) \equiv u(\alpha)u(\beta)u(\alpha+\beta)$ where $u : \Phi_g \to \{\pm 1\}$ is an arbitrary function with $u(\alpha) = u(-\alpha)$. It would therefore appear that there are $2^{|\Phi_g^+|}$ possible normalisations. In fact there are only $2^{|\Phi_g^+| - \mathrm{rank}\, g}$ as some choices of u give equivalent normalisations.

These non–equivalent normalisations are in one–to–one correspondence with the choice of signs for the $N_{\beta,\gamma}$ when we have singled out a *unique* decomposition of each positive non–simple root into the sum of an extraspecial pair of roots, $\alpha = \beta + \gamma$, (pp 58-60 [28]). An ordered pair of roots (α,β) is said to be **special** when

<u>*(i)*</u> $\alpha + \beta \in \Phi_g$ and $\alpha > \beta > 0$,

and **extraspecial** when

<u>*(i)*</u> (α,β) is special,

<u>*(ii)*</u> $\forall$ special pairs (γ,δ) such that $\alpha+\beta = \gamma+\delta$ we have $\beta \le \delta$.

In summary;

<u>Theorem (1.2) :</u> Given a Chevalley basis (or any basis in which $|\ N_{\alpha,\beta}\ |$ is given and we have chosen $N_{\alpha,\beta} = \pm N_{-\alpha,-\beta}$) we can choose the signs of the strucure constants $N_{\alpha,\beta}$ for all the extraspecial pairs of roots (α,β) and then the structure constants for all other pairs of roots are determined. There are $2^{\frac{1}{2}(\dim g - 3\,\mathrm{rank}\, g)}$ such normalisations.

However for a given normalisation there are still an infinite number of Chevalley bases because we can always change to a new Chevalley basis with the same normalisation by mapping $E_\alpha \mapsto \eta(\alpha)E_\alpha$, where $\eta : \Phi_g \rightarrow S^1$ is a homomorphism, *i.e.* $\eta(\alpha)\eta(\beta) = \eta(\alpha+\beta)$.

The Jacobi identity and restrictions on the structure constants.

We get restrictions on the structure constants whenever we consider a Jacobi identity containing three step operators,

$$[E_\alpha,[E_\beta,E_\gamma]] + [E_\beta,[E_\gamma,E_\alpha]] + [E_\gamma,[E_\alpha,E_\beta]] = 0.$$

By looking at all the truly different choices of step operators we can choose to make we find that there are up to five possible restrictions on the structure constants. We summarize the results for each algebra in the following table.

Table 1.3 : *Jacobi restrictions on the* $N_{\alpha,\beta}$.

	$\alpha+\beta$	$\beta+\gamma$	$\alpha+\gamma$	$\alpha+\beta+\gamma$	A,D,E.	B	G	F	C
1	$\times$	$\surd$	$\surd$	$\surd$	Y	Y	Y	Y	Y
2	$\surd$	$\surd$	$\surd$	$\surd$	N	N	N	Y	Y
3	$\times$	$\surd$	0	$\surd$	Y (1)	Y (1)	Y	Y (1)	Y (1,2)
4	$\surd$	$\surd$	0	$\surd$	N	Y (0)	Y (±1)	Y (0)	Y (0)
5	$\surd$	$\surd$	$\surd$	0	Y	Y	Y	Y	Y

Key :

A $\surd$ in the $\alpha+\beta$ column denotes that $\alpha+\beta \in \Phi_g$.

A $\times$ in the $\alpha+\beta$ column denotes that $\alpha+\beta \notin \Phi_g$.

A 0 in the $\alpha+\beta$ column denotes that $\alpha+\beta = 0$.

A Y/N denotes whether this restriction holds or not for a given algebra and the number in brackets denotes the possible values of $\alpha.\beta$ if it does.

$$1: N_{\beta,\gamma}N_{\beta+\gamma,\alpha} = N_{\gamma,\alpha}N_{\beta,\alpha+\gamma} \qquad \textbf{2 - cocycle condition} \tag{1.16}$$

$$2: N_{\alpha,\beta+\gamma}N_{\beta,\gamma} + N_{\beta,\gamma+\alpha}N_{\gamma,\alpha} + N_{\gamma,\alpha+\beta}N_{\alpha,\beta} = 0 \tag{1.17}$$

$$3: N_{\alpha,\beta-\alpha}N_{\beta,-\alpha} = -B_\alpha \alpha.\beta \tag{1.18}$$

$$4: N_{\alpha,\beta-\alpha}N_{\beta,-\alpha} + N_{-\alpha,\alpha+\beta}N_{\alpha,\beta} = -B_\alpha \alpha.\beta \tag{1.19}$$

$$5: N_{\beta,-(\alpha+\beta)}B_\alpha = N_{\alpha,\beta}B_{\alpha+\beta} = N_{-(\alpha+\beta),\alpha}B_\beta \tag{1.20}$$

Simply – laced algebras and cocycles.

A choice of structure constants must satisfy (1.12) and an appropriate combination from (1.16) to (1.20).

Theorem (1.3) : For simply laced algebras a suitable set of structure constants is given by a subset of the 2-cocycle $\varepsilon : \Lambda_R \times \Lambda_R \to \{\pm 1\}$ if we take,

$$\varepsilon(\alpha, -\alpha) = B_\alpha \tag{1.21}$$

$$\varepsilon(\alpha, 0) = 1 \tag{1.22}$$

$$\varepsilon(\alpha, \beta) = (-1)^{\alpha.\beta}\varepsilon(\beta, \alpha) \tag{1.23}$$

$\forall \ \alpha, \beta \in \Phi_g$ and choose either,

$$\text{(I)}\ B_\alpha = 1 \quad \& \quad \varepsilon(\alpha,\beta) = -\varepsilon(-\alpha,-\beta) \quad \forall\, \alpha,\beta \in \Phi_g \tag{1.24}$$

$$\text{(II)}\ B_\alpha = -1 \quad \& \quad \varepsilon(\alpha,\beta) = \varepsilon(-\alpha,-\beta) \quad \forall\, \alpha,\beta \in \Phi_g \tag{1.25}$$

Proof : We must show that ε satisfies (1.12), (1.16), (1.18) and (1.20).

(i) If $\alpha + \beta \in \Phi_g$ then $\alpha.\beta = -1$, therefore from (1.23) ε satisfies (1.12).

(ii) ε is a 2-cocycle by definition and therefore satisfies (1.16).

<u>*(iii)*</u>

$$\begin{aligned}\varepsilon(\alpha,\beta-\alpha)\varepsilon(\beta,-\alpha) &= -\varepsilon(-\alpha,\alpha)\varepsilon(\beta,0) \text{ as } \varepsilon \text{ is a } 2-\text{cocycle on } \Lambda_R. \\ &= -\mathrm{B}_\alpha.1\end{aligned}$$

<u>*(iv)*</u>

$$\begin{aligned}\varepsilon(\beta,-(\alpha+\beta))\mathrm{B}_\alpha &= \frac{\varepsilon(-\beta,\beta)}{\varepsilon(-\alpha,-\beta)}\varepsilon(-\alpha,0)\mathrm{B}_\alpha \text{ as } \varepsilon \text{ is a } 2-\text{cocycle on } \Lambda_R. \\ &= -\mathrm{B}_\alpha\mathrm{B}_\beta\varepsilon(-\alpha,-\beta) \\ &= \mathrm{B}_{\alpha+\beta}\varepsilon(\alpha,\beta) \text{ if we have either (I) or (II).}\end{aligned}$$

Similarly: $\varepsilon(-(\alpha+\beta),\alpha)\mathrm{B}_\beta = \varepsilon(\alpha,\beta)\mathrm{B}_{\alpha+\beta}$.

<u>Note :</u> In particular the choice **(II)** is consistent with the restriction that ε is a **bilinear function** *i.e.*

$$\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma).$$

(1.23) then implies,

$$\varepsilon(\gamma,\alpha+\beta) = \varepsilon(\gamma,\alpha)\varepsilon(\gamma,\beta).$$

Notice that in fact bilinearity implies that $\varepsilon(\alpha,\beta) = \varepsilon(-\alpha,-\beta)$ as,

$$\begin{aligned}\varepsilon(\alpha,\beta)\varepsilon(-\alpha,\beta) &= \varepsilon(0,\beta) = 1, \\ \varepsilon(-\alpha,-\beta)\varepsilon(-\alpha,\beta) &= \varepsilon(-\alpha,0) = 1.\end{aligned}$$

Following [2] we can construct such an ε explicitly. Let α_i be simple roots of g, and A$=$ (a_{ij}) its Cartan matrix.

(i) Define

$$\varepsilon(\alpha_i,\alpha_j) = \begin{cases} +1 & \text{if } i<j, \\ -1 & \text{if } i=j, \\ (-1)^{a_{ji}} & \text{if } i>j. \end{cases}$$

(ii) Extend this definition to the whole lattice by bimultiplicity *i.e.*

$$\varepsilon\left(\sum_{i=1}^{\mathrm{rank}\,g} n_i\alpha_i, \sum_{j=1}^{\mathrm{rank}\,g} n_j\alpha_j\right) = \prod_{\substack{i=1\\j=1}}^{\mathrm{rank}\,g} \varepsilon(\alpha_i,\alpha_j)^{n_in_j}.$$

It can be shown that ε satisfies (1.12), (1.22), (1.23) and

$$\varepsilon(\alpha,-\alpha) = \varepsilon(\alpha,\alpha) = (-1)^{\frac{\alpha^2}{2}},$$

which is consistent with $\mathrm{B}_\alpha = -1 \;\; \forall\, \alpha \in \Phi_{\mathrm{g}}$. Thus (1.21) and **(II)** are also satisfied.

Notice that as pointed out in [2], it is not possible to assume bimultiplicity and $\varepsilon(\alpha,\alpha) = 1$ (or $\mathrm{B}_\alpha = 1$) on all the roots as stated in [21].

This choice coincides with the one given in [22] where $\sigma : \Lambda_R \times \Lambda_R \to \mathbb{Z}_2$ is a bilinear form such that

$$\sigma(\alpha,\beta) + \sigma(\beta,\alpha) = \alpha.\beta \pmod 2$$

and $\varepsilon(\alpha,\beta) \equiv (-1)^{\sigma(\alpha,\beta)}$. We have taken

$$\sigma(\alpha_i,\alpha_j) = \begin{cases} 0 & i < j, \\ 1 & i = j, \\ a_{ji} \pmod 2 & i > j. \end{cases}$$

1.2.2. Hermiticity.

Consider a representation R of g on the canonical n-dimensional *unitary space* $\mathrm{V} \cong \mathbb{C}^n$, $\mathrm{R}:\mathrm{g} \to \mathrm{GL}_n(\mathbb{C})$ such that

(i) $\mathrm{R}([x,y]) = \mathrm{R}(x)\mathrm{R}(y) - \mathrm{R}(y)\mathrm{R}(x) \quad \forall\, x,y \in \mathrm{g}$,

(ii) $\mathrm{R}(\lambda x) = \lambda \mathrm{R}(x) \quad \forall\, \lambda \in \mathbb{C},\; x \in \mathrm{g}$.

Every linear operator on V has an associated adjoint operator obtained by hermitian conjugation. The set of operators $\{-\mathrm{R}^\dagger(x) \mid x \in \mathrm{g}\}$ form another, **conjugate representation** of g on V.

We now wish to examine the possible hermiticity properties of such a representation. In particular we look for a mapping $^+ : \mathrm{g} \to \mathrm{g}$ such that

$$\mathrm{R}(x^+) = \lambda_x \mathrm{R}^\dagger(x). \tag{1.26}$$

Lemma (1.4) : The mapping $^+ : g \to g$ satisfying (1.26) must also satisfy;

$$\underline{(1)}\ [x,y]^+ = \frac{\lambda_{[x,y]}}{\lambda_x\lambda_y}[y^+,x^+] \quad \forall\, x,\, y\, \in g,$$
$$\underline{(2)}\ (x^+)^+ = x \qquad \forall\, x \in g,$$
$$\underline{(3)}\ \lambda_x^+ \doteq \lambda_x^* \qquad \forall\, \lambda_x \in \mathbb{C},$$
$$\underline{(4)}\ \lambda_x\lambda_y = \lambda_{xy} \text{ and } (xy)^+ = y^+x^+ \quad \forall\, \lambda_x,\, \lambda_y \in \mathbb{C},\ x,\, y \in g.$$

Proof :

(1)

$$\begin{aligned} R^\dagger([x,y]) &= \{R(x)R(y) - R(y)R(x)\}^\dagger, \\ &= R^\dagger(y)R^\dagger(x) - R^\dagger(x)R^\dagger(y). \end{aligned}$$

Therefore (1.26) implies;

$$\begin{aligned} R([x,y]^+) &= \frac{\lambda_{[x,y]}}{\lambda_x\lambda_y}\left\{R(y^+)R(x^+) - R(x^+)R(y^+)\right\}. \\ &= \frac{\lambda_{[x,y]}}{\lambda_x\lambda_y}R([y^+,x^+]). \end{aligned}$$

Thus $[x,y]^+ = \frac{\lambda_{[x,y]}}{\lambda_x\lambda_y}[y^+,x^+]$.

(2)

$$\begin{aligned} (R^\dagger)^\dagger(x) &= R(x), \\ &= R^\dagger\left(\frac{1}{\lambda_x}x^+\right), \\ &= \frac{1}{\lambda_x\lambda_{x^+}^+}R\left(((x)^+)^+\right). \end{aligned}$$

Therefore $(x^+)^+ = x$, and $\lambda_x\lambda_{x^+}^+ = 1$.

(3)

$$\begin{aligned} R^\dagger(\lambda x) &= \lambda^* R^\dagger(x), \\ &= \lambda_x R(\lambda^* x^+), \\ &= \lambda_x R\left((\lambda x)^+\right). \end{aligned}$$

Therefore $\lambda^+ = \lambda^*\ \forall\,\lambda$, *i.e.* $^+$ acts as complex conjugation on the scalars. Thus from (2) $\lambda_{x^+}^*\lambda_x = 1$.

(4)

$$
\begin{aligned}
\mathrm{R}^{\dagger}(xy) &= \lambda_{xy}\mathrm{R}\left((xy)^{+}\right), \\
&= \{\mathrm{R}(x)\mathrm{R}(y)\}^{\dagger}, \\
&= \mathrm{R}(y)^{\dagger}\mathrm{R}(x)^{\dagger}, \\
&= \lambda_x\lambda_y\mathrm{R}(y^{+})\mathrm{R}(x^{+}), \\
&= \lambda_x\lambda_y\mathrm{R}(y^{+}x^{+}).
\end{aligned}
$$

Therefore $\lambda_{xy} = \lambda_x\lambda_y$ and $(xy)^{+} = y^{+}x^{+}$.

For any Lie algebra we can always choose a basis, h_i, for a given CSA, H, such that

$$\mathrm{R}^{+}(\mathrm{h}_i) = \lambda_i\mathrm{R}(\mathrm{h}_i) \quad \text{and } \lambda_i = \mathrm{e}^{i\phi_i},$$

i.e. a basis in which the $\mathrm{R}(\mathrm{h}_i)$ are *simultaneously* diagonalised. This is because R(H) is a commuting family of semisimple endomorphisms as H is abelian. In addition for a complex Lie algebra we can rescale this basis, $\mathrm{h}_i \mapsto \mathrm{e}^{-i\frac{\phi_i}{2}}\mathrm{h}_i$ so that all the matrices, $\mathrm{R}(\mathrm{h}_i)$, are hermitian. With respect to this basis,

$$\mathrm{h}_i^{+} = \mathrm{h}_i.$$

The restrictions that this, the commutation relations and Lemma (1.4) places on E_α^{+} and the structure constants is given in the following theorem

Theorem (1.5) : If $\mathrm{B}_\alpha (= \mathrm{B}_{-\alpha}) = N_\alpha \mathrm{e}^{i\theta_\alpha}$, $N_\alpha \in \mathbf{R}^{+}$ and $\mathrm{H}^{+} = \mathrm{H}$ then;

1. $\mathrm{E}_\alpha^{+} = \mathrm{e}^{-i\theta_\alpha}\mathrm{E}_{-\alpha}$,
2. $\mathrm{N}_{-\beta,-\alpha} = \mathrm{N}_{\alpha,\beta}^{*}\mathrm{e}^{i(\theta_\alpha+\theta_\beta-\theta_{\alpha+\beta})}$.

Proof : Let $\mathrm{E}_\beta^{+} \equiv \sum_{\gamma\in\Phi_g} \lambda_\gamma \mathrm{E}_{-\gamma} + \delta.\mathrm{H}$.

(1.10) gives
$$\sum_{\gamma\in\Phi_g} \lambda_\gamma[\mathrm{E}_{-\gamma},\mathrm{H}_i] = \beta_i\left(\sum_{\gamma\in\Phi_g} \lambda_\gamma \mathrm{E}_{-\gamma} + \delta.\mathrm{H}\right).$$

Therefore $E_\beta^+ = \lambda_\beta E_{-\beta}$.

(1.11) gives
$$\lambda_\alpha \lambda_\beta [E_{-\beta}, E_{-\alpha}] = \begin{cases} N^*_{\alpha,\beta}\lambda_{\alpha+\beta}E_{-(\alpha+\beta)} & \alpha+\beta \in \Phi_g; \\ 0 & \alpha+\beta \notin \Phi_g; \\ B^*_\alpha \alpha.H & \alpha+\beta = 0. \end{cases}$$

Thus

$$\lambda_\alpha \lambda_\beta N_{-\beta,-\alpha} = N^*_{\alpha,\beta}\lambda_{\alpha+\beta}, \tag{1.27}$$
$$\text{and} \quad \lambda_\alpha \lambda_{-\alpha} B_\alpha = B^*_\alpha \quad \text{therefore} \quad \lambda_\alpha \lambda_{-\alpha} = e^{-2i\theta_\alpha} \tag{1.28}$$

$$(E_\alpha^+)^+ = E_\alpha \quad \text{gives} \quad \lambda^*_\alpha \lambda_{-\alpha} = 1 \tag{1.29}$$

Thus $\lambda_\alpha = r_\alpha e^{-i\theta_\alpha}$, $\lambda_{-\alpha} = \frac{1}{r_\alpha} e^{-i\theta_\alpha}$ but (1.27) implies that $r_\alpha r_\beta = r_{\alpha+\beta}$ so take $r_\alpha = 1 \ \forall\, \alpha \in \Phi_g$.

If we now wish to restrict g to be a real Lie algebra then all the structure constants, $N_{\alpha,\beta}$, and the B_αs must be real, so $\theta_\alpha = 0, \pi \ \forall\, \alpha \in \Phi_g$. We have a bewildering variety of different possible choices and conventions. Summarizing we have the following choices if we wish all the step operators to have the same hermiticity properties.

<u>Table 1.3 :</u> *Summary of conventions.*

$E_\alpha^+ =$	$N_{\alpha,\beta} = -N_{-\alpha,-\beta}$	$N_{\alpha,\beta} = N_{-\alpha,-\beta}$
$E_{-\alpha}$ ($B_\alpha \in \mathbf{R}^+$)	(I) $\mathbf{R}$	$i\mathbf{R}$
$-E_{-\alpha}$ ($B_\alpha \in \mathbf{R}^-$)	$i\mathbf{R}$	(II) $\mathbf{R}$

The entries in the boxes show whether the structure constants must be purely real or purely imaginary. For a real form of the Lie algebra we must choose a convention corresponding to a shaded box. Within these two general conventions it is further possible to choose the B_α so that we have a Chevalley basis (*i.e.* $N_{\alpha,\beta} \in \mathbb{Z}$). The two choices (I) and (II) of p 18 are marked. We can see from Table 1.3 that case (I) corresponds to the usual hermiticity conditions on the generators.

2. An Introduction to Strings.

2.1 THE BOSONIC STRING

Classical String :

Consider a closed string moving in a D–dimensional Minkowski space $\mathbf{R}^{D-1,1}$. As it does so it will sweep out a worldsheet $X^\mu(\sigma,\tau)$ where $\sigma \in [0,2\pi]$, $\tau \in (-\infty,\infty)$, $\mu = 1,.....,D$ and we have $X^\mu(0,\tau) = X^\mu(2\pi,\tau)$.

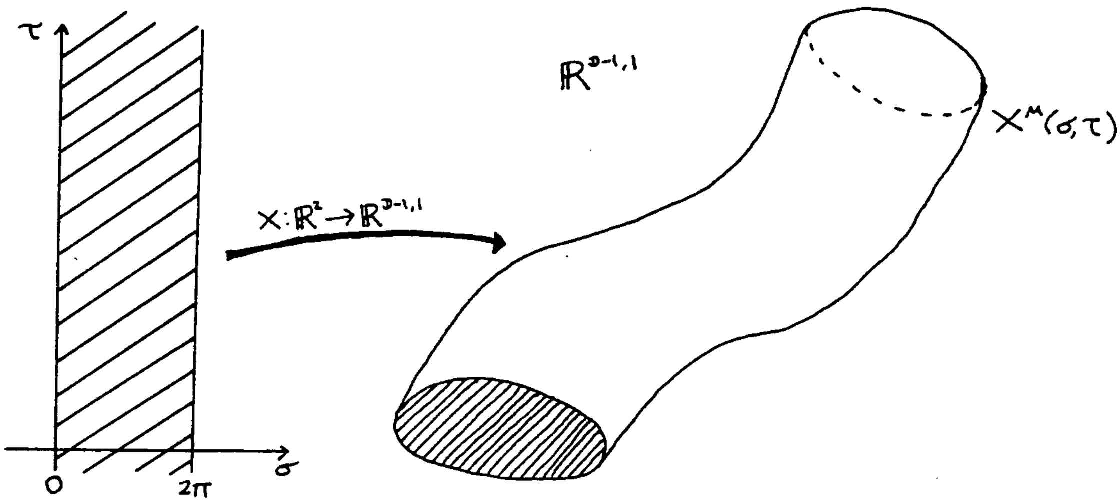


Fig 2.1 : Worldsheet of a propagating closed string in $\mathbf{R}^{D-1,1}$

The area of the worldsheet is assumed to be a minimal surface in $\mathbf{R}^{D-1,1}$ and thus the motion of the string is governed by the action [30,31],

$$S = \frac{1}{T}\int d\sigma d\tau \sqrt{(\partial_\tau X.\partial_\sigma X)^2 - (\partial_\tau X)^2(\partial_\sigma X)^2}$$

where the **string tension**, T, is a constant having the dimensions of a force per unit length and $\partial_\tau X = \frac{\partial X}{\partial \tau}$, $\partial_\sigma X = \frac{\partial X}{\partial \sigma}$. We can use the reparametrisation invariance of the worldsheet to choose an orthogonal co–ordinate system on the worldsheet,

$$\partial_\tau X.\partial_\sigma X = 0, \qquad (2.1)$$

$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0. \qquad (2.2)$$

With respect to this co-ordinate system the equations of motion are simply D copies of the **wave equation,**

$$(\partial_{\tau\tau} - \partial_{\sigma\sigma})\mathrm{X}^\mu = 0.$$

They have the general solution

$$\mathrm{X}^\mu(\sigma,\tau) = \mathrm{X}^\mu_{\mathrm{L}}(\tau+\sigma) + \mathrm{X}^\mu_{\mathrm{R}}(\tau-\sigma)$$

where $\mathrm{X}^\mu_{\mathrm{L}}(\tau+\sigma)$ consists of the **left movers** whilst $\mathrm{X}^\mu_{\mathrm{R}}(\tau-\sigma)$ consists of the **right movers.** Explicitly,

$$\begin{aligned}
\mathrm{X}^\mu_{\mathrm{L}}(\tau+\sigma) &= \mathrm{q}^\mu_{\mathrm{L}} + \mathrm{p}^\mu_{\mathrm{L}}(\tau+\sigma) + \frac{i}{2}\sum_{\mathrm{n}\neq 0}\frac{\mathrm{a}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{e}^{-i\mathrm{n}(\tau+\sigma)},\\
\mathrm{X}^\mu_{\mathrm{R}}(\tau-\sigma) &= \mathrm{q}^\mu_{\mathrm{R}} + \mathrm{p}^\mu_{\mathrm{R}}(\tau-\sigma) + \frac{i}{2}\sum_{\mathrm{n}\neq 0}\frac{\overline{\mathrm{a}}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{e}^{-i\mathrm{n}(\tau-\sigma)},
\end{aligned}$$

with $\mathrm{a}^{\mu\,\dagger}_{\mathrm{n}} = \mathrm{a}^\mu_{-\mathrm{n}}$, $\overline{\mathrm{a}}^{\mu\,\dagger}_{\mathrm{n}} = \overline{\mathrm{a}}^\mu_{-\mathrm{n}}$. If we define $\mathrm{a}^\mu_0 \equiv \mathrm{p}^\mu_{\mathrm{L}}$, $\overline{\mathrm{a}}^\mu_0 \equiv \mathrm{p}^\mu_{\mathrm{R}}$ then the constraints become;

$$\mathrm{p}^\mu_{\mathrm{L}} = \mathrm{p}^\mu_{\mathrm{R}}, \tag{2.3}$$

$$\mathrm{L}_{\mathrm{n}} \equiv \frac{1}{2}\sum_{\mathrm{m}\in \mathrm{Z}} \mathrm{a}_{\mathrm{m}}\mathrm{a}_{\mathrm{n-m}} = 0 \quad \forall\, \mathrm{n} \in \mathbb{Z} \tag{2.4}$$

$$\overline{\mathrm{L}}_{\mathrm{n}} \equiv \frac{1}{2}\sum_{\mathrm{m}\in \mathrm{Z}} \overline{\mathrm{a}}_{\mathrm{m}}\overline{\mathrm{a}}_{\mathrm{n-m}} = 0 \quad \forall\, \mathrm{n} \in \mathbb{Z} \tag{2.5}$$

Thus

$$\mathrm{X}^\mu(\sigma,\tau) = \mathrm{q}^\mu + \mathrm{p}^\mu\tau + \frac{i}{2}\sum_{\mathrm{n}\neq 0}\left\{\frac{\mathrm{a}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{e}^{-i\mathrm{n}(\tau+\sigma)} + \frac{\overline{\mathrm{a}}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{e}^{-i\mathrm{n}(\tau-\sigma)}\right\}$$

where $\mathrm{q}^\mu \equiv \mathrm{q}^\mu_{\mathrm{L}} + \mathrm{q}^\mu_{\mathrm{R}}$ and $\mathrm{p}^\mu \equiv \mathrm{p}^\mu_{\mathrm{L}} + \mathrm{p}^\mu_{\mathrm{R}}$.

The **Fubini–Veneziano fields** are defined by

$$\begin{aligned}
\mathrm{Q}^\mu(\mathrm{z}) &\equiv \mathrm{q}^\mu - i\mathrm{p}^\mu \ln \mathrm{z} + i\sum_{\mathrm{n}\neq 0}\frac{\mathrm{a}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{z}^{-\mathrm{n}},\\
\overline{\mathrm{Q}}^\mu(\mathrm{z}) &\equiv \mathrm{q}^\mu - i\mathrm{p}^\mu \ln \mathrm{z} + i\sum_{\mathrm{n}\neq 0}\frac{\overline{\mathrm{a}}^\mu_{\mathrm{n}}}{\mathrm{n}}\mathrm{z}^{-\mathrm{n}},
\end{aligned}$$

so that

$$X^{\mu}(\sigma,\tau) = \frac{1}{2}\left\{Q^{\mu}(e^{i(\tau+\sigma)}) + \overline{Q}^{\mu}(e^{i(\tau-\sigma)})\right\}.$$

Let us introduce the **momentum density**

$$P^{\mu}(\sigma,\tau) = \frac{\partial X^{\mu}}{\partial \tau}(\sigma,\tau).$$

If we let $P^{\mu}(z) = iz\frac{dQ^{\mu}}{dz} = \sum_{n\in Z} a_n^{\mu} z^{-n}$, $\overline{P}^{\mu}(z) = iz\frac{d\overline{Q}^{\mu}}{dz} = \sum_{n\in Z} \overline{a}_n^{\mu} z^{-n}$, then

$$P^{\mu}(\sigma,\tau) = \frac{1}{2}\left\{P^{\mu}(e^{i(\tau+\sigma)}) + \overline{P}^{\mu}(e^{i(\tau-\sigma)})\right\}.$$

Quantum String :

To *first quantise* the string we replace the co-ordinates a_n^{μ}, $\overline{a}_n^{\mu}$, p^{μ}, and q^{μ} by operators and impose the canonical **covariant equal time quantisation condition**,

$$[X^{\mu}(\sigma,\tau), P^{\nu}(\sigma',\tau)] = i\hbar\delta(\sigma-\sigma')g^{\mu\nu}.$$

This means the harmonic oscillators must satisfy the following commutation relations

$$[a_m^{\mu}, a_n^{\nu}] = m g^{\mu\nu}\delta_{m+n,0},$$
$$[q^{\mu}, p^{\nu}] = i g^{\mu\nu}.$$

The operation of complex conjugation becomes hermitian conjugation so that $a_n^{\mu\,\dagger} = a_{-n}^{\mu}$, $\overline{a}_n^{\mu\,\dagger} = \overline{a}_{-n}^{\mu}$, $p^{\mu\,\dagger} = p^{\mu}$, and $q^{\mu\,\dagger} = q^{\mu}$. The oscillators a_m^{μ}, $\overline{a}_m^{\mu}$ are **creation operators** if $n, m < 0$ and **annihilation operators** if $n, m > 0$. The string Hilbert space can be written as

$$H = F_L \otimes F_R \otimes V_{R^D},$$

where $F_L\,/\,F_R$ is the Fock space representation of the Heisenberg algebra spanned by the L / R oscillators and the identity operator and V_{R^D} is the infinite–dimensional space spanned by the momentum eigenstates. The momentum eigenstates are generated from the **vacuum state**, $\mid 0\rangle$, by the position operator as follows,

$$\mid \gamma\rangle = e^{i\gamma.q} \mid 0\rangle.$$

Thus a typical element of the Hilbert space H can be written as,

$$\prod_{i=1}^{r}\prod_{j=1}^{s} a_{-n_i}^{\mu_i}\bar{a}_{-m_j}^{\nu_j} e^{i\gamma.q} \mid 0\rangle.$$

The constraints (2.4) and (2.5) now become *operator constraints* on the states of the model. We define

$$L_n \equiv \frac{1}{2}\sum_{n\in Z} : a_m a_{n-m} :,$$
$$\overline{L}_n \equiv \frac{1}{2}\sum_{n\in Z} : \bar{a}_m \bar{a}_{n-m} :,$$

where the double dots : : denotes a normal ordering with respect to the mode index. This is necessary to eliminate the ambiguity in the ordering of the oscillators in L_0 and $\overline{L}_0$ to make them well defined.

$$: a_n a_m := \begin{cases} a_n a_m & m > n, \\ a_m a_n & m < n. \end{cases}$$

For $\mid \Phi\rangle$ to be a **physical state** it must satisfy

$$L_n \mid \Phi\rangle = \overline{L}_n \mid \Phi\rangle = 0 \quad n > 0, \tag{2.6}$$
$$(L_0 - \overline{L}_0) \mid \Phi\rangle = 0, \tag{2.7}$$
$$(L_0 + \overline{L}_0 - \lambda) \mid \Phi\rangle = 0, \tag{2.8}$$

where λ is an arbitrary constant arising from the ambiguity in the choice of normal ordering. (2.8) is known as the **mass shell condition**. It can be shown that if the theory is to be Lorentz invariant, entirely transverse and ghost free, that is no negative norm states couple to physical states, then we must have $D = 26$ and $\lambda = 2$ (see for example the review in [32]). This means the lowest mass state is a **tachyon**.

The two sets of constraint operators separately satisfy the **Virasoro algebra**,

$$\begin{aligned}
[\mathrm{L_m}, \mathrm{L_n}] &= (\mathrm{m}-\mathrm{n})\mathrm{L_{m+n}} + \frac{\mathrm{D}}{12}\mathrm{m}(\mathrm{m}^2-1)\delta_{\mathrm{m+n},0}, \\
[\overline{\mathrm{L}}_\mathrm{m}, \overline{\mathrm{L}}_\mathrm{n}] &= (\mathrm{m}-\mathrm{n})\overline{\mathrm{L}}_{\mathrm{m+n}} + \frac{\mathrm{D}}{12}\mathrm{m}(\mathrm{m}^2-1)\delta_{\mathrm{m+n},0}.
\end{aligned}$$

2.2 THE HETEROTIC STRING:

It was discovered a couple of years ago that a perfectly consistent *closed orientable* string theory can be constructed by taking the left movers of a 26–dimensional closed bosonic string and adding them to the right movers of a 10–dimensional closed supersymmetric string [17,18]. This chiral hybrid construction was called the **heterotic string.** It is possible because the physical degrees of freedom of closed strings are 2–dimensional free fields that can be separated into left and right movers. These never mix, not even in the presence of string interactions as long as only orientable world–sheets are considered. This is due to the fact that closed string interactions are constructed order by order in perturbation theory by modifying the topology of the strings world–sheet. Thus in terms of the first quantised 2–dimensional theory no interactions are thereby introduced and the right and left movers still propagate freely and independently of each other [17,33]. The resulting string theory is inherently chiral, anomaly free, Lorentz invariant and $\mathrm{N}=1$ supersymmetric in $\mathrm{D}=10$.

The extra 16–dimensional left moving co–ordinates are treated as *internal dimensions* and compactified onto a space T. This space was originally thought to have to be a flat compact manifold, that is a 16–torus,

$$\mathrm{T} = \frac{\mathbf{R}^{16}}{\Gamma},$$

where Γ is a 16–dimensional lattice. In fact more general spaces are allowable as we shall see later.

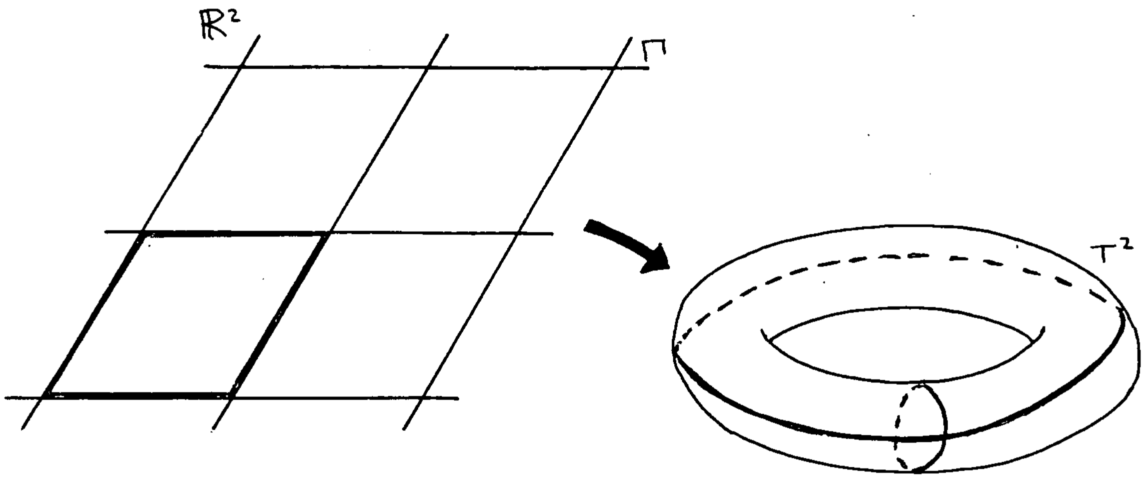


Fig 2.2 : Compactification of $\mathbf{R}^2$ onto T^2 via a lattice Γ

The fact that a closed string has no distinguished point means that Γ must be an **even** lattice, whilst examination of the 1–loop scattering amplitudes implies that Γ must be **self–dual** if we are to have a consistent theory of interacting closed strings. There are only two such lattices in $\mathbf{R}^{16}$, the root lattice of $E_8 \times E_8$ and a lattice consisting of the root lattice of so(32) plus one set of the spinor weights of $\frac{\text{spin}(32)}{Z_2}$.

The co–ordinates for the internal space have the following normal–mode expansion,

$$X^i(\tau+\sigma) = q^i + p^i\tau + L^i\sigma + i\sum_{n\neq 0}\frac{a_n^i}{n}e^{-in(\tau+\sigma)} \qquad i = 1, \ldots\ldots, 16,$$

where $L \in \Gamma$ is a **winding vector** describing how the closed string is wound around the torus T. Because the string is purely left moving the allowed winding vectors must correspond to allowed momenta.

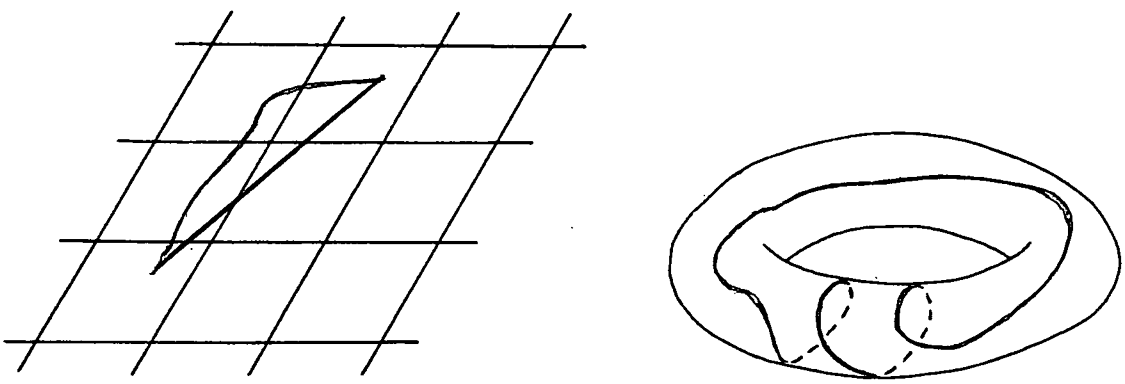


Fig 2.3 : A string configuration corresponding to a non – trivial winding vector.

Non–zero values of L correspond to **winding sectors** of the resulting string theory. Strings with a non–vanishing L are topologically stable and are therefore looked on as **soliton states**. They exist because of the multiple–connectedness of the configuration space and the extended nature of the strings and have no analogues in point particle theories. They can be created in pairs from a string with L $=$ 0 and therefore must be included in the full theory to preserve *modular invariance.*

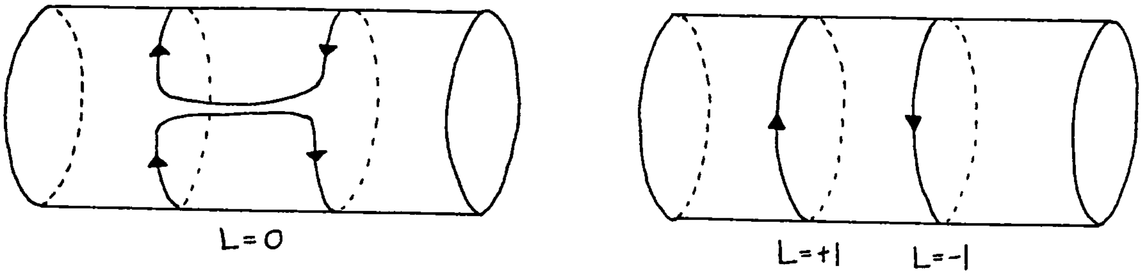


Fig 2.4 : The creation of 2 string states with non – zero winding numbers from a state with L $=$ 0.

When we quantise this theory we form the correct Hilbert space as follows.

1. Firstly we consider the Hilbert space corresponding to strings propagating in the original covering space.
2. Then we consider all the states which correspond to strings in this covering space that close up to a lattice vector of Γ *i.e.*

$$X^{i} = X^{i} + L^{i} \qquad \text{where } L \in \Gamma.$$

3. Finally we project onto the Hilbert subspace of states that are invariant under the lifting of the action corresponding to the translation by a lattice vector.

$$e^{2\pi i L.p} \mid Physical\rangle = \mid Physical\rangle.$$

Thus in particular the allowed centre of mass momenta are restricted to lie on the dual lattice Γ^{*}.

We can introduce an interaction between these closed strings via the **vertex operators** which are used in the calculation of scattering amplitudes. For the heterotic string such a vertex operator can be decomposed into two parts. A supersymmetric string vertex that acts in the 10 space–time dimensions and a piece that acts in the 16 internal dimensions which we shall denote by $\mathrm{V}(\phi, \mathrm{z})$. If we take $\mathrm{z} = \mathrm{e}^{i(\tau+\sigma)}$ and $\mathrm{L} = \mathrm{p}$ then,

$$\mathrm{X}(\mathrm{z}) = \mathrm{q} - i\mathrm{p}\ln \mathrm{z} + i \sum_{\mathrm{n}\neq 0} \frac{\mathrm{a_n}}{\mathrm{n}} \mathrm{z}^{-\mathrm{n}}$$

and

$$\mathrm{V}(\phi, \mathrm{z}) \equiv : \epsilon_1.\partial_{\mathrm{z}}^{n_1}\mathrm{X}(\mathrm{z}).....\epsilon_r.\partial_{\mathrm{z}}^{n_r}\mathrm{X}(\mathrm{z})\mathrm{e}^{i\alpha.\mathrm{X}(\mathrm{z})} : \mathrm{C}_\alpha$$

where $\alpha \in \Gamma$. Such vertex operators correspond to **conformal fields** with conformal dimension $\sum_{i=1}^{r} n_i + \frac{\alpha^2}{2}$. We let $\mathrm{N} = \sum_{i=1}^{r} n_i$. The physical states are in one–to–one correspondence with the asymptotic limits as $\mathrm{z} \rightarrow 0$ of the vertex operators acting on the vacuum state,

$$\epsilon_1.\mathrm{a}_{-\mathrm{n}_1}.....\epsilon_r.\mathrm{a}_{-\mathrm{n}_r} \mid \alpha\rangle \ \leftrightarrow \ \mid \phi\rangle \equiv \lim_{\mathrm{z}\rightarrow 0} \mathrm{V}(\phi, \mathrm{z}) \mid 0\rangle.$$

The total left moving states are created by the full vertex operator

$$\mathrm{V_L}(\phi', \mathrm{z}) = \mathrm{V}_{10}(\phi'', \mathrm{z})\mathrm{V}(\phi, \mathrm{z}).$$

For physical states this full vertex operator must have conformal dimension one. For massless states this can be achieved by the conformal weight of $\mathrm{V}_{10}(\phi'', \mathrm{z})$ equaling one in which case the resulting physical states form part of a $\mathrm{D} = 10$, $\mathrm{N} = 1$ supergravity multiplet. The other possibility is that the conformal weight of $\mathrm{V}(\phi, \mathrm{z})$ equals one. This occurs if $\mathrm{N} = 0$, $\alpha^2 = 2$ or $\mathrm{N} = 1$, $\alpha^2 = 0$. These physical states form a $\mathrm{D} = 10$, $\mathrm{N} = 1$ super–Yang–Mills multiplet of G where $\mathrm{G} = \mathrm{E}_8 \times \mathrm{E}_8$ or $\frac{\mathrm{spin}(32)}{Z_2}$ depending on the choice of Γ.

The moments of the vertex operators with conformal dimension one (*i.e.* $\mathrm{N} = 0$, $\alpha^2 = 2$ or $\mathrm{N} = 1$, $\alpha^2 = 0$) as we shall show later give via the Frenkel–Kac construction [21,22,2,26] a representation of the Kac–Moody algebra $\hat{\mathrm{g}}$, the affinisation of the Lie algebra g whose root lattice is Γ. Thus the Fock space of physical states forms a representation space for $\hat{g}$ and each mass level forms a representation of g. In this case the gradation of the representation corresponds to the mass spectrum of the physical states.

Despite its nice features there are a number of problems with the heterotic string as a model of the real world. One is the rather large symmetry group of the massless states, $E_8 \times E_8$ or $\frac{\mathrm{spin}(32)}{Z_2}$ and another is the unphysical number of dimensions. One way of solving the latter problem, was given in [34]. The idea was to compactify $\mathbf{R}^{9,1}$ onto $\mathbf{R}^{3,1} \otimes K$, where K was a **Calabi–Yau manifold**, a Ricci–flat Kähler manifold with SU(3) holonomy, which gives chiral fermions in four dimensions. However Calabi–Yau manifolds are complicated, metrics for them are difficult to find and calculations of the resulting interactions in the compactified theory are very hard.

One way of avoiding this problem is to replace K by an approximation to a Calabi–Yau manifold such as an orbifold. An **orbifold** Ω is the quotient space formed by dividing a manifold M by the action of a discrete group G,

$$\Omega \equiv \frac{\mathrm{M}}{\mathrm{G}}.$$

If G acts freely on M then the resulting orbifold Ω is a smooth manifold. However if the action of G has fixed points then Ω will have singularities corresponding to these fixed points. If the original manifold is flat then the orbifold will also be flat everywhere except at the singularities where the curvature blows up. Some orbifolds, such as the *Z–orbifold* [34] can be used to construct Calabi–Yau spaces by removing the singularities and replacing them by appropriate manifolds. It was soon seen that strings propagating on orbifolds actually gave perfectly consistent string theories in their own right [14,15].

As well as considering an orbifold compactification of the physical space–time dimensions of the theory we can also consider an orbifold compactification of the internal degrees of freedom. Indeed this gives us a method of reducing the symmetry of the massless states without having to replace the lattice on which the original toroidal compactification is done and thus the theory remains consistent. This gives a method of **spontaneous symmetry breaking** for the string as the vacuum vector of such a theory does not share the full symmetry of the theory. We examine a general orbifold construction in more detail in the next section.

2.3 ORBIFOLD COMPACTIFICATION AND TWISTED STRINGS.

Let us forget the particular physical restrictions that are placed on a string theory such as the heterotic string and look at some general constructions. Let g be a simply laced algebra with root lattice Λ_R and rank g $=$ d. We shall just consider the general twisted string theories on an orbifold obtained by quotienting $\mathbf{R}^{\mathrm{d}}$ by $\Lambda_R \rtimes \mathrm{W}$ where W is in general a non-abelian subgroup of aut Φ_{g}. In particular we shall be interested in the case when $\mathrm{W} \cong \mathbb{Z}_n$ is an abelian group generated by one element $\sigma \in$ aut Φ_{g} of order n. We shall denote this group by $\langle\sigma\rangle$.

Classical String :

Let G be a **space group** of the form

$$\mathrm{G} = \Lambda_R \rtimes \mathrm{W}$$

where $\mathrm{W} <$ aut Φ_{g} and Λ_R is the group of lattice translations. Λ_R is a normal subgroup of G. An element $\mathrm{g} = (\alpha, \sigma)$ of G has the following action on $\mathbf{R}^{\mathrm{d}}$,

$$\mathrm{g}(\beta) = \sigma(\beta) + \alpha.$$

The orbifold Ω is obtained by dividing $\mathbf{R}^{\mathrm{d}}$ by the action of G, that is identifying all the points that are in the same orbit of G. This explains the choice of name. If α_i are the simple roots of Λ_R then we define $\mathrm{U} \subset \mathbf{R}^{\mathrm{d}}$ to be the **fundamental cell** of Λ_R given by

$$\mathrm{U} \equiv \{\mathrm{n}_i\alpha_i \mid \mathrm{n}_i \in [0,1)\}.$$

We identify U with the torus T^{d}. All points in $\mathbf{R}^{\mathrm{d}}$ can be translated into U by the addition of an element of Λ_R. Let Π be the projection from $\mathbf{R}^{\mathrm{d}}$ onto T^{d} such that $\Pi(\alpha) = \bar{\alpha} \in \mathrm{U}$ where $\bar{\alpha} = \alpha \bmod \Lambda_R$. W, which is called the **point group**, has a well defined action on the torus

$$\mathrm{T}^{\mathrm{d}} = \frac{\mathbf{R}^{\mathrm{d}}}{\Lambda_R}$$

given by $\bar{\sigma} \equiv \Pi \circ \sigma \circ \Pi^{-1}$ where $\Pi^{-1} : \mathrm{T}^{\mathrm{d}} \to \mathrm{U}$ maps the torus into the fundamental cell. Equivalently $\bar{\sigma} = \frac{G}{\Lambda_R}$. Thus

$$\Omega \cong \frac{\mathbf{R}^{\mathrm{d}}}{\Lambda_R \rtimes \mathrm{W}} \cong \frac{\mathrm{T}^{\mathrm{d}}}{\overline{\mathrm{W}}}.$$

W is the **holonomy group** of Ω [15]. It is discrete, in comparison to a smooth manifold which has a continuous holonomy group.

Let us in particular take $\sigma \in \operatorname{aut} \Phi_{\mathrm{g}}$ to be an element of order n. The twisted string boundary conditions are then

$$\bar{\mathrm{X}}(2\pi, \tau) = \bar{\sigma}\left(\bar{\mathrm{X}}(0, \tau)\right).$$

Consider the complexification V of $\mathbf{R}^{\mathrm{d}}$. We can diagonalise σ in V,

$$\sigma = \mathrm{diag}(\omega^{n_1}, \ldots.., \omega^{n_d})$$

where ω is a primitive n^{th} root of unity. If we denote the m^{th} eigenspace of V by $\mathrm{V}_m = \{x \in \mathrm{V} \mid \sigma(x) = \omega^m x\}$ then we have the following eigenspace decomposition of V,

$$\mathrm{V} = \bigoplus_{m=0}^{n-1} \mathrm{V}_m.$$

Note that some of the V_m might be empty. Let $\mathrm{P}_m : \mathrm{V} \to \mathrm{V}_m$ such that $\mathrm{P}_m = \sum_{r=0}^{n-1} \omega^{-mr}\sigma^r$ be the projection operator onto the m^{th} eigenspace and let $x_m \equiv \mathrm{P}_m(x)$ for $x \in \mathrm{V}$.

If we consider the motion of a purely left–moving closed string on Ω we can see that the m^{th} eigenspace of the normal–mode decomposition of X(z) can be written as,

$$\mathrm{X}_m(\mathrm{z}) = \mathrm{q}_m - i\mathrm{p}_0 \ln \mathrm{z}\,\delta_{m,0} + i \sum_{r \in \mathrm{Z}+\frac{m}{n}} \frac{\mathrm{a}_r}{r}\mathrm{z}^{-r}.$$

In the case of $m = 0$; p_0, $\mathrm{q}_0 \in \mathrm{V}_0$ and we just have the equation of a closed string moving on a smaller torus of $\dim \mathrm{V}_0$, namely the torus obtained by compactifying on $\Lambda_R^0 \equiv \mathrm{V}_0 \cap \Lambda_R$.

The points $q = \sum_{m=0}^{n} q_m$ correspond to points of V which project onto points $\bar{q}$ in T that are fixed under $\bar{\sigma}$, *i.e.* $\bar{\sigma}(\bar{q}) = \bar{q}$, so

$$q \in M_\sigma \equiv \{x \in \mathbf{R}^d \,|\, (1-\sigma)x \in \Lambda_R\}.$$

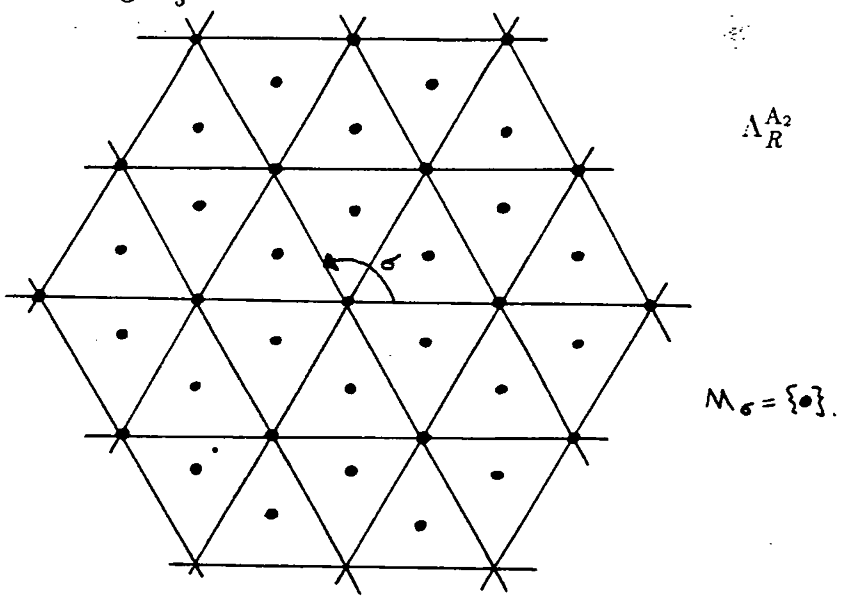
The points $\bar{q}$ are called **orbifold fixed point singularities** and the set of such fixed points is denoted by $T^{\bar{\sigma}} = \Pi(M_\sigma)$. In fact they are only points if $\dim P_0V = \{0\}$, otherwise they consist of $\dim P_0V$–dimensional subspaces of the orbifold. We label these subspaces by $\bar{q}_f$ where

$$q_f \in \frac{M_\sigma}{P_0 V}\,.$$

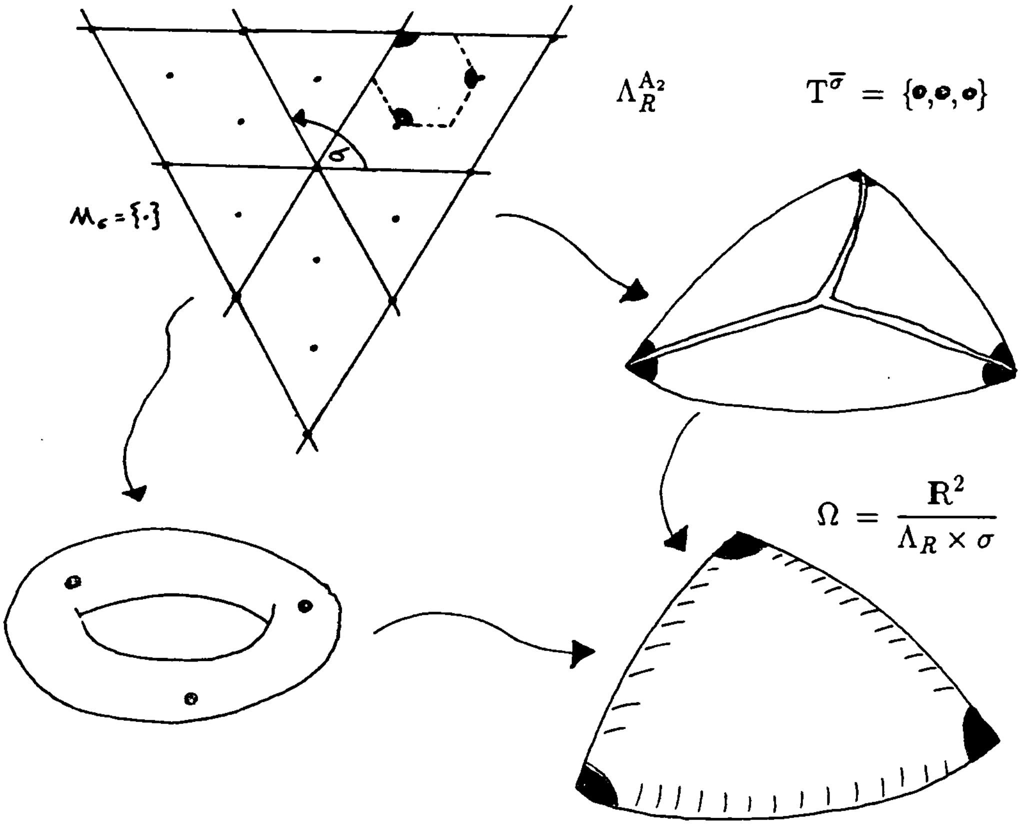
The number of singular subspaces is given by

$$F \equiv \left|\Pi\left(\frac{M_\sigma}{P_0V}\right)\right| = \left|\frac{T^{\bar{\sigma}}}{\Pi\,(P_0V)}\right|\,.$$

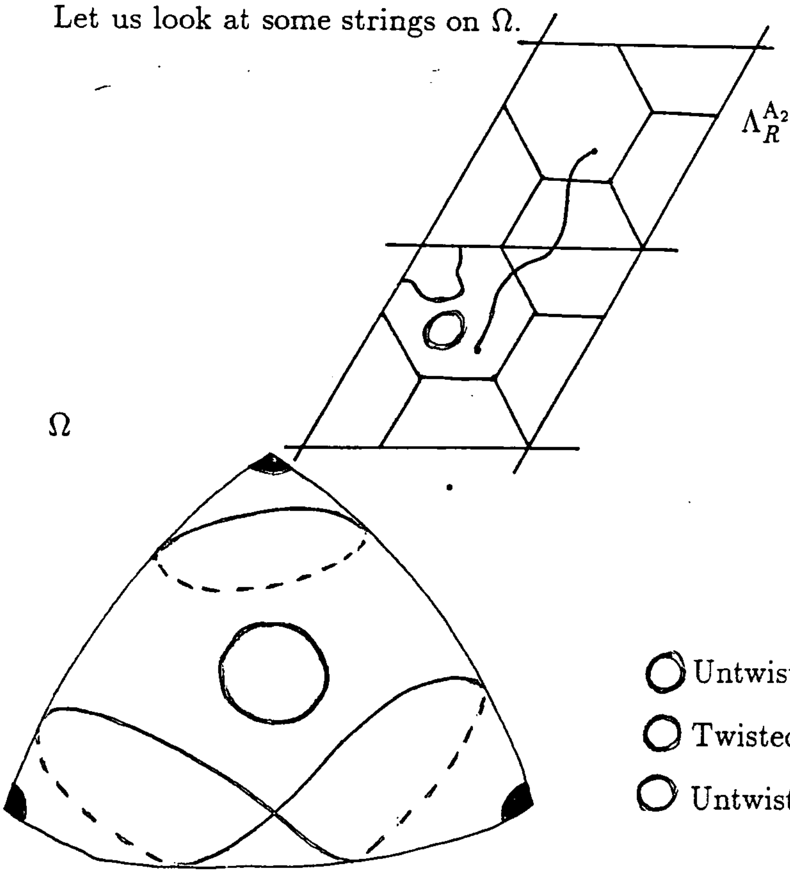
Let us look at simple two–dimensional example. Take $g = A_2$ and σ to be the rotation through $\frac{2\pi}{3}$.



$M_\sigma = \Lambda_R \oplus (w_1 + \Lambda_R) \oplus (w_2 + \Lambda_R)$, where w_1 and w_2 are the fundamental weights of A_2.



Let us look at some strings on Ω.



○ Untwisted string L = 0.

○ Twisted string.

○ Untwisted string with L = 1.

Quantum string :

The procedure for constructing the Hilbert space for the theory of quantised strings on an orbifold is just a slight generalisation of that given in the previous section for strings on a torus.

1. Take the Hilbert space, H_M^e, for strings propagating on the covering manifold M.
2. As we are dealing with a closed string theory we must also consider strings that only close up to an element of G *i.e.*

$$X(2\pi) = gX(0) \qquad \text{where } g \in G.$$

Notice that such boundary conditions mean that the centre of mass of the string must sit at a fixed point or in a fixed subspace. Thus for each $g \in G$ there is a new sector of the Hilbert space H_M^g in which the boundary conditions are changed to periodicity up to a transformation by g. These are generalisations of the winding sectors. Let $H_M = \bigoplus_{g \in G} H_M^g$.

3. Finally we must project each H_M^g onto a Hilbert space H_0^g that is invariant under the action of $\overline{G}$, where $\overline{G}$ is a representation of G on H_M^g. Notice that if G is non–abelian then $\overline{G}$ will mix up the twisted sectors H_M^g corresponding to elements in the same conjugacy class. For example assume that we have a state in H_M^g, corresponding to the boundary conditions,

$$X(2\pi) = gX(0).$$

Acting on the state by an element $h \in G$ gives

$$hX(2\pi) = hgh^{-1}[hX(0)].$$

So this new string is periodic up to hgh^{-1} and therefore an element of $H_M^{hgh^{-1}}$. The symmetry group of each H_M^g is thus only equal to its **centralizer** (or little group) $C(g) \equiv \{g' \in G \mid g'g = gg'\}$ which is a subgroup of G. To form $\overline{G}$ invariant states we must project each sector H_M^g onto its $\overline{C(g)}$ invariant subspace $H_M^{C(g)}$ and then sum over all the corresponding states in different $H_M^{C(g')}$ where g' is conjugate to g.

Thus in the final theory there is a sector for each *conjugacy class* of G. The ones corresponding to conjugacy classes other than the identity are known as **twisted sectors.** This construction is necessary if the string Hilbert space is to be G invariant and modular invariant [14]. If G is abelian then the conjugacy classes are all one–dimensional and the construction reduces to projecting each H_M^g onto its $\overline{G}$ invariant subspace.

It is interesting to note that we can take $G = \Lambda_R \rtimes \mathrm{aut}\, \Phi_g$ to produce a string model with a sector corresponding to each inequivalent vertex operator representation of $g^{(\tau)}$.

We now return to the case when $W = \langle \sigma \rangle$. In particular we look at the first twisted sector, H_M^σ, corresponding to string states which only close up to σ.

For each eigenspace V_m we introduce an orthonormal basis e_m^i, where $i = 1, ...,$ $\dim V_m$. The oscillators are then given by a_r^i with $r \in \mathbb{Z} + \frac{m}{n}$ and $i = 1, ..., \dim V_m$. Canonical quantisation gives the following commutation relations,

$$\begin{aligned} [q_0^i, p_0^j] &= i\delta^{ij}, \\ [a_r^i, a_s^j] &= r\delta^{ij}\delta_{r+s,0}. \end{aligned}$$

The Hilbert space is of the form

$$H^\sigma = F^\sigma \otimes V^\sigma$$

where F^σ is a Fock space for the oscillators and V^σ is the zero–mode Hilbert space describing the centre of mass of the string. It in turn consists of two parts,

$$V^\sigma = \mathbf{C}(P_0\Lambda_R) \otimes \overline{V}^\sigma.$$

The first part is the complex span of the momentum eigenstates of the form $| \alpha_0 \rangle$ where $\alpha_0 \in P_0\Lambda_R$. This is because quantising the string implies that the momenta lie on the lattice $(\Lambda_R^0)^* = P_0\Lambda_W$. In fact we only wish to choose momentum from the sublattice, $P_0\Lambda_R \subset P_0\Lambda_W$. The second part is less obvious. It is not given by the naive guess of taking states corresponding to fixed subspaces of the form $| \bar{q_f} \rangle$. It is in fact [5,16] an irreducible projective representation of $\frac{M_\sigma}{M'_\sigma}$ where

$$M'_\sigma \equiv \{x \in M_\sigma \mid \Psi(x,y) = 1 \; \forall \; y \in M_\sigma\}$$

and $\Psi : M_\sigma \times M_\sigma \to \mathbb{C} - \{0\}$ is the alternating bimultiplicative form given by,

$$\begin{aligned}\Psi(x,y) &\equiv e^{2\pi i x.(1-\sigma)y}, \\ &= C(\alpha,\beta).\end{aligned}$$

where $\alpha = (1-\sigma)x + \alpha_0$ ($\alpha_0 \in P_0\Lambda_R$), $\beta = (1-\sigma)y$ and $C(\alpha,\beta)$ is the commutator map of [5]. This is both necessary for the Hilbert space to be a representation space for a Kac–Moody algebra [5,16] and required by modular invariance as we only have left moving modes [14,35]. We will discuss it in more detail in Chapter 6.

We define the **momentum field** by, $P(z) \equiv iz\partial_z X(z)$. The *conformal group* is generated by moments of the **Virasoro field,**

$$L(z) = \sum_{n \in Z} L_n z^{-n} \equiv \frac{1}{2} : P^2(z) : .$$

The moments L_n generate a copy of the Virasoro algebra with L_0 shifted by

$$\eta = \frac{1}{4n^2} \sum_{i=1}^{\mathrm{rank}\, g} n_i(n_i - n).$$

Thus the **conformal weight of the vacuum** is η as

$$L_0 \mid 0\rangle = \eta \mid 0\rangle.$$

L_0 gives a gradation of the twisted Hilbert space,

$$H^\sigma = \bigoplus_{r \in \frac{1}{2n}Z} H^\sigma_r.$$

The **partition function** defined by $P(q) \equiv \sum_{r \in \frac{1}{2n}Z} (\dim H^\sigma_r) q^r$ is,

$$P(q) = c_\sigma \frac{\sum_{\alpha \in P_0\Lambda_R} q^{\frac{1}{2}\alpha^2 + \eta}}{\prod_{m=1}^{\infty} (1 - q^{\frac{m}{n}})^{d\,(m \bmod n)}}$$

where c_σ is the degeneracy of the vacuum and $d\,(m \bmod n)$ is the dimension of the eigenspace with the eigenvalue ω^m.

The **twisted vertex operators** for this sector are of the form

$$V(\phi,z) \equiv N_\sigma(\alpha) z^{\frac{-(\alpha^2-\alpha_0^2)}{2}} : i\epsilon_1.\partial_z^{n_1} X_\sigma(z).....i\epsilon_r.\partial_z^{n_r} X_\sigma(z) e^{i\alpha.X_\sigma(z)} : C_\sigma(\alpha)$$

where $\alpha \in \Lambda_R$, $N_\sigma(\alpha)$ is a normalisation factor and $C_\sigma(\alpha)$ is a cocycle operator on the zero–mode space that we will explain more fully in Chapter 6. The overall factor of $z^{\frac{-(\alpha^2-\alpha_0^2)}{2}}$ is necessary to make $V(\phi,z)$ a conformal field. The twisted vertex operator describes the emission of an untwisted string state from a twisted string.

The twisted vertex operators of conformal dimension one give a representation of a Kac–Moody algebra $g^{(\tau)}$ associated with the Lie algebra g. Again this will be shown explicitly in Chapter 6.

Such a construction has a different gradation from the untwisted sector which means the mass spectrum is altered. Let g_0 be the Lie subalgebra of $g^{(\tau)}$ that commutes with L_0,

$$[L_0, g_0] = 0.$$

Thus states at each mass level now form representations of g_0 and in particular the massless states of the resulting theory are in the adjoint representation of g_0. The symmetry of the theory is therefore broken from g to g_0. There is a slight caveat to this remark in that sometimes the various sectors can combine together to restore the original symmetry [15, 36]. It is not clear, at least to the author, what conditions are necessary for this to occur.

For the case of $W = \langle\sigma\rangle$ the other sectors of the theory are obtained by replacing σ by a power of σ in the previous working. In general each sector of the twisted string gives a different graded representation of $g^{(\tau)}$ where τ might vary from sector to sector if σ is outer. Note that the twisted sectors $H_M^{\sigma^m}$ and $H_M^{\sigma^{n-m}}$ have the same vacuum degeneracy and the same invariant subalgebra g_0 but correspond to conjugate representation spaces of $g^{(\tau)}$. In general all the sectors $H_M^{\sigma^m}$ where m is relatively prime to n have the same vacuum degeneracy as each other and the same invariant subalgebra as the untwisted sector when it has been projected onto $\overline{G}$ invariant states. In these cases we do not have to make any further projection of $H_M^{\sigma^m}$ onto $\overline{G}$ invariant states as σ^m generates $\langle\sigma\rangle$. If we take σ to be of prime order then all the sectors fall into this category.

It can be seen from the form of various of the equations that a lot of things are simplified if σ leaves no directions fixed *i.e.* det(1-σ) $\neq$ 0, as then $P_0\Lambda_R = \{0\}$. If this is to the case for all the sectors of the theory then we must have det(1-σ^m) $\neq$ 0 for $m = 1, \ldots\ldots, n-1$. This motivates a study of such σ, which we call **no fixed point automorphisms** or **NFPAs**. In Chapter 5 we determine all the NFPAs of simple Lie algebra root systems. On NFPA orbifolds the momentum is eliminated in all the twisted sectors. As in this case the centre of mass variables are not dynamical the twisted strings sit at, and oscillate about the orbifold fixed points.

3. Realizations of the Kac-Moody algebras.

In this chapter we will show how to realize an arbitrary integer gradation of the infinite dimensional Kac–Moody algebras in terms of the finite dimensional Lie algebras. This is done by realizing them as central extensions of loop algebras and subalgebras of them. It is basically an exposition of work found in [1]. To start with we need to know something about the automorphisms of Lie algebras.

3.1 Automorphisms of Lie algebras.

An **automorphism** of a Lie algebra, g, is a one–to–one mapping of g onto itself that preserves the operation of commutation. That is S:g→g such that

$$S([x,y]) = [S(x),S(y)] \qquad \forall x,y \in g.$$

The collection of all such automorphisms form a group denoted by **aut g**. It has a normal subgroup **int g**, consisting of automorphisms which are generated by the action of elements of g. Such automorphisms are called **inner automorphisms**, the rest, aut g–int g, are known as **outer automorphisms**. If $a \in g$ then the corresponding inner automorphism, $S_a : g \to g$, is given by

$$S_a(x) = e^a x e^{-a} \qquad \forall x, \in g. \tag{3.1}$$

Firstly it should be noted that in (3.1) x and a are considered both as elements of g and the compact group, G, associated with g for the implied multiplication to make any sense. Secondly we need to check that the resulting element $S_a(x)$ is in fact an element of g. On explicit evaluation it can be seen that,

$$\begin{aligned} x \mapsto S_a(x) &= x + [a,x] + \frac{1}{2!}[a,[a,x]] + \ldots. \\ &= e^{ad(a)}x \in g \end{aligned}$$

where ad(a) is the **adjoint representation** of a, that is the linear map, ad(a):g→g, such that,

$$\mathrm{ad(a)(x)} = [\mathrm{a,x}] \qquad \forall \mathrm{x} \in \mathrm{g}$$

It is easy to see that int g is isomorphic to g as $S_a(S_b(x)) = S_{ab}(x)$. As int g is a normal subgroup of aut g we can form the **factor group**

$$\Gamma_g = \frac{\mathrm{aut\,g}}{\mathrm{int\,g}}.$$

For simple Lie algebras Γ_g is always a finite group corresponding to the symmetry group of the Lie algebra's Dynkin diagram.

Table 3.1 : *Factor groups for the simple Lie algebras.*

g	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
Γ_g	$\mathbb{Z}_2$	1	1	$\mathbb{Z}_2$ $n \neq 4$ S_3 $n = 4$	$\mathbb{Z}_2$	1	1	1	1

All inner automorphisms of g can be written in the form

$$\begin{aligned} S_a(H) &= H, \\ S_a(E_\alpha) &= e^{2\pi i \chi.\alpha} E_\alpha, \end{aligned}$$

after an appropriate choice of CSA, where χ lies in the fundamental Weyl chamber, $c(\Delta)$, of the root system of g, Φ_g. χ is known as a **shift vector.**

For a purely outer automorphism, A, corresponding to a Dynkin diagram symmetry, $X{:}\Lambda_R \rightarrow \Lambda_R$ we have

$$\begin{aligned} A(H) &= X(H), \\ A(E_\alpha) &= \epsilon_\alpha^{(X)} E_{X(\alpha)}, \end{aligned}$$

where $\epsilon_\alpha^{(X)} = \pm 1$ such that

$$\begin{aligned}\epsilon_\alpha^{(X)}\epsilon_\beta^{(X)} &= \frac{N_{\alpha,\beta}}{N_{X(\alpha),X(\beta)}}\epsilon_{\alpha+\beta}^{(X)} \\ \epsilon_\alpha^{(X)} &= \epsilon_{-\alpha}^{(X)}\end{aligned}$$

In addition the $\epsilon_\alpha^{(X)}$ can be chosen equal to 1 on the simple roots (see [28] p 201). We call such a Lie algebra automorphism a **diagram automorphism.**

In general, as we shall see later, we can extend any root system automorphism to an algebra automorphism. The order of this algebra automorphism may be equal to *or* double the order of the root system automorphism, depending on the particular choice of automorphism.

Lemma (3.1) : Except for the diagram automorphism of A_{2n} we may choose the $\epsilon_\alpha^{(X)}$ so that

$$\frac{N_{X(\alpha),X(\beta)}}{N_{\alpha,\beta}} = 1 \;\; \forall\, \alpha,\beta \in \Phi_g.$$

Proof : Recall that the choice of normalisation of the structure constants is only unique up to multiplication by a function $f(\alpha,\beta) = U_\alpha U_\beta U_{\alpha+\beta}$ where $U : \Phi_g \to \{\pm 1\}$ is an arbitrary function with $U_{-\alpha} = U_\alpha$ (see Subsection 1.2.1. p 21). Thus after a renormalisation we have

$$\frac{f(\alpha,\beta)N_{\alpha,\beta}}{f\left(X(\alpha),X(\beta)\right)N_{X(\alpha),X(\beta)}} = \frac{\epsilon_\alpha^{(X)}\epsilon_\beta^{(X)}}{\epsilon_{\alpha+\beta}^{(X)}}$$

and hence

$$\frac{N_{X(\alpha),X(\beta)}}{N_{\alpha,\beta}} = \epsilon_\alpha^{(X)}\epsilon_\beta^{(X)}\epsilon_{\alpha+\beta}^{(X)}\frac{f(\alpha,\beta)}{f\left(X(\alpha),X(\beta)\right)}.$$

(i) If the extension of an order two Dynkin diagram automorphism is also of order two then,

$$\epsilon_{X(\alpha)}^{(X)}\epsilon_\alpha^{(X)} = 1 \quad \forall\, \alpha \in \Phi_g. \tag{3.2}$$

In this case all we have to do is to choose the $U_\alpha \;\forall\, \alpha \in \Phi_g^+$ such that

$$U_\alpha = \epsilon_\alpha^{(X)}U_{X(\alpha)}, \tag{3.3}$$

$$U_{-\alpha} = U_{\alpha}. \tag{3.4}$$

This is consistent because

$$U_{X(\alpha)} = \epsilon^{(X)}_{X(\alpha)} U_{\alpha} \tag{3.5}$$

reduces to (3.3) by the use of (3.2). In addition we have no problem with assigning U_{α} separately on the positive and negative roots so as to satisfy (3.4) because X does not mix them.

We have a problem however if the order of X has to be doubled when it is extended to an algebra automorphism as then (3.3) and (3.5) give

$$U_{\alpha} U_{X(\alpha)} = \epsilon^{(X)}_{\alpha} \epsilon^{(X)}_{X(\alpha)} U_{X(\alpha)} U_{\alpha}$$

but we must have $\epsilon^{(X)}_{\alpha} \epsilon^{(X)}_{X(\alpha)} = -1$ for some $\alpha \in \Phi_{g}$.

The only case when the order of a diagram automorphism is twice that of the corresponding Dynkin diagram automorphism is for A_{2n}.

(ii) Similiarly for the third order diagram of D_4, whose order is not doubled, we can consistently take,

$$U_{\alpha} = \epsilon^{(X)}_{\alpha} U_{X(\alpha)}, \qquad \forall\, \alpha \in \Phi^{+}_{g}$$
$$U_{-\alpha} = U_{\alpha}.$$

The structure of the full automorphism group is that of a semi-direct product,

$$\mathrm{aut}\, g = \mathrm{int}\, g \rtimes \Gamma_{g}.$$

3.2 Subgroup of finite order automorphisms.

The subset of automorphisms of a Lie algebra, g, of finite order form a subgroup as the product of two finite order automorphisms is also a finite order automorphism. We shall denote this subgroup by $\mathbf{aut}^{F}\mathbf{g}$ and its normal subgroup of finite order inner automorphisms by $\mathbf{int}^{F}\mathbf{g}$.

Consider a finite order automorphism, $\Sigma \in \mathrm{aut}^{\mathrm{F}}\,\mathrm{g}$, of order N, of a complex Lie algebra g. If $\omega = e^{\frac{2\pi i}{N}}$ then we can decompose g into its eigenspaces under Σ,

$$\mathrm{g} = \bigoplus_{\mathrm{n} \in \mathbb{Z}_N} \mathrm{g}_\mathrm{n},$$

where $\mathrm{g}_\mathrm{n} = \{x \in \mathrm{g} \mid \Sigma(x) = \omega^\mathrm{n} x\}$ is the eigenspace of Σ with eigenvalue ω^n. This gives a $\mathbb{Z}_N$ gradation of g as

$$[\mathrm{g}_\mathrm{m}, \mathrm{g}_\mathrm{n}] \subset \mathrm{g}_{\mathrm{m}+\mathrm{n}} \qquad \mathrm{m}, \mathrm{n} \in \mathbb{Z}_N \tag{3.6}$$

Conversely we can form a finite order automorphism from a gradation of g by decomposing any element into its components with respect to this gradation and then multiplying each component by its appropriate eigenvalue.

The fixed eigenspace g_0 can be seen from (3.6) to form a closed subalgebra, known as the **invariant subalgebra**. This is in fact true whether Σ is of finite order or not. In addition the other eigenspaces form representations of g_0 as

$$[\mathrm{g}_0, \mathrm{g}_\mathrm{n}] \subset \mathrm{g}_\mathrm{n} \qquad \mathrm{n} \in \mathbb{Z}_N. \tag{3.7}$$

3.2.1 Invariant subalgebras for diagram automorphisms.

Let g be a simple Lie algebra with Chevalley generators $e_i^{\pm} (= E_{\pm\alpha_i})$ and a CSA basis $h_i (= \alpha_i.H)$ where i=1,...,rank g . Consider a diagram automorphism of order τ,

$$\begin{aligned} H &\mapsto X(H) \\ E_{\pm\alpha_i} &\mapsto E_{\pm X(\alpha_i)} \end{aligned}$$

where we have chosen the $\epsilon_\alpha^{(X)} = 1$ on the simple roots. Thus let us introduce a permutation $\chi \in S_{\mathrm{rank}\,\mathrm{g}}$ such that $\chi(i) = j$ iff $X(\alpha_i) = \alpha_j$. Notice that under X a simple root is either fixed or has order τ. Let O(g,τ) be the number of orbits of simple roots under X, and let $[i]$ denote the orbit containing the simple root α_i.

Table 3.2 : *Diagram automorphisms.*

$\tau = 1$	$O(g,1) = \text{rank}\, g$	$\chi(i) = i$
$\tau = 2$	$O(A_{2n},2) = n$ $O(A_{2n-1},2) = n$ $O(D_n,2) = n\text{-}1$ $O(E_6,2) = 4$	$\chi(i) = 2n - i$ $\chi(i) = (2n-1) - i$ $\chi(i) = i,\ 1 \leq i \leq n-2,\ \ \chi(n) = n-1$ $\chi(1) = 5, \chi(2) = 4, \chi(3) = 3, \chi(6) = 6$
$\tau = 3$	$O(D_4,3) = 2$	$\chi(1) = 3, \chi(3) = 4, \chi(4) = 1, \chi(2) = 2$

We define the following elements of g_0,

$$\begin{array}{llll} E_i^{\pm} = e_i^{\pm} & H_i = h_i & \chi(i) = i \\ E_i^{\pm} = e_i^{\pm} + \ldots + e_{\chi^{\tau}(i)}^{\pm} & H_i = h_i + \ldots + h_{\chi^{\tau}(i)} & \chi^{\tau}(i) = i \end{array}$$

Except for A_{2n}, $\tau = 2$ where we set,

$$\begin{array}{llll} E_i^{\pm} = e_i^{\pm} + e_{\chi(i)}^{\pm} & H_i = h_i + h_{\chi(i)} & i \neq n \\ E_n^{\pm} = \sqrt{2}(e_n^{\pm} + e_{n+1}^{\pm}) & H_n = (h_n + h_{n+1}) & i = n \end{array}$$

Let us choose the following indexing for these elements, to correspond with the labeling of the roots of g_0,

$$\begin{array}{rlll} A_n, D_n : & E_i^{\pm} = E_{[i]}^{\pm} & 1 \leq i \leq O(g,\tau) & \\ E_6 : & E_i^{\pm} = E_{[i]}^{\pm} & 1 \leq i \leq 3 & E_4 = E_{[6]} \end{array}$$

X gives a $\mathbb{Z}_{\tau}$ gradation of g, *i.e.*

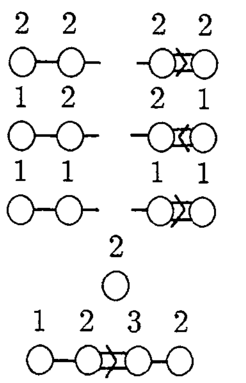
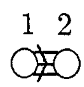
$$\begin{array}{lll} g = g_0 & \tau = 1 \\ g = g_0 \oplus g_1 & \tau = 2 \\ g = g_0 \oplus g_1 \oplus g_2 & \tau = 3 \end{array}$$

Where g_1 is an irreducible representation of g_0 and the representation g_2 is equivalent to that of g_1.

The set of elements $E_i^{\pm}$ $1 \leq i \leq O(g,\tau)$, are Chevalley generators of g_0 whilst the elements H_i $1 \leq i \leq O(g,\tau)$ form a basis for the invariant CSA, H_0.

Let Θ_0 be the (unique) highest weight of the representation g_1. (Take $\Theta_0 = \Theta$, the highest root of g, for $\tau = 1$). If we denote the simple roots of g_0 by α_i, $1 \leq i \leq O(g,\tau)$, and $\Theta_0 = \sum_{i=1}^{O(g,\tau)} a_i\alpha_i$ then the values of a_i are given in Table 3.3.

Table 3.3 : *Invariant subalgebras and highest weights of* g_1 *representations.*

<table>
<tr><th>τ</th><th>g</th><th>g_0</th><th>$D(g_0)$ & a_i</th></tr>
<tr><td>2</td><td>A_{2n}, $n \geq 2$</td><td>B_n</td><td rowspan="5">
</td></tr>
<tr><td>2</td><td>A_{2n-1}, $n \geq 3$</td><td>C_n</td></tr>
<tr><td>2</td><td>D_n, $n \geq 2$</td><td>B_{n-1}</td></tr>
<tr><td>2</td><td>A_2</td><td>A_1</td></tr>
<tr><td>2</td><td>E_6</td><td>F_4</td></tr>
<tr><td>3</td><td>D_4</td><td>G_2</td><td>
</td></tr>
</table>

Notice that by adding the root $-\Theta_0$ to the Dynkin diagram of g_0 we get the corresponding Dynkin diagram of the twisted Kac-Moody algebra $g^{(\tau)}$. Hence $a_i = k_i^\tau$ for $i = 1, ..., O(g,\tau)$.

If we choose $E_0^{\pm} = E_{\mp\Theta_0}$, $H_0 = -\Theta_0$, then for $\tau > 1$ the elements E_i^+, $i = 1, ..., O(g,\tau)$ generate the Lie algebra g [1]. Thus we take $\alpha_0 = -\Theta_0$ and see that $\sum_{i=0}^{O(g,\tau)} k_i^\tau \alpha_i = 0$, where the k_i^τ are the Kac labels for the appropriate (un)twisted Kac-Moody algebra.

3.2.2 Classification of all finite order automorphisms.

We now state a couple of results concerning the classification of all finite order automorphisms of simple Lie algebras and a method of determining the associated invariant subalgebras which are both due to Kac. We refer the reader to [25,1] for the proofs of these results and to [37] for a thorough exposition. We use the notation introduced in the previous section.

Theorem (3.2) : Classification of finite order automorphisms : Let g be a simple Lie algebra, n = rank g and s = $(s_0, \ldots\ldots, s_{O(g,\tau)})$ be a sequence of non-negative relatively prime integers. Set

$$N = \tau \sum_{i=0}^{O(g,\tau)} k_i^\tau s_i .$$

Then:

(i) We define an automorphism of g by,

$$\sigma_{s;\tau}(E_j) \equiv e^{\frac{2\pi i S_j}{N}} E_j \quad j = 0, \ldots\ldots, O(g,\tau). \tag{3.8}$$

We call it the **automorphism of type (s;τ).**

(ii) The automorphisms $\sigma_{s;\tau}$ exhaust all the N^{th} order automorphisms of g, up to conjugation by an automorphism of g.

(iii) The elements $\sigma_{s;\tau}$ and $\sigma_{s';\tau'}$ are conjugate by an automorphism of g iff $k = k'$ and the sequences s and s' can be transformed into each other by an automorphism of the Dynkin diagram $D(g^\tau)$.

(iv) τ is the least positive integer for which ${\sigma_{s;\tau}}^{\tau}$ is an inner automorphism.

Proof : See [1] pp 96-98.

Corollary(3.3) : Let w_i, $1 \leq i \leq \text{rank}\, g$, be the fundamental weights of a simple Lie algebra g and X a symmetry of its Dynkin diagram of order τ. In general, if $(g,\tau) \neq (A_{2n}, 2)$, we may rewrite (3.8) as

$$H \mapsto X(H),$$
$$E_\alpha \mapsto e^{2\pi i \alpha.\delta} E_{X(\alpha)},$$

where $\delta = \frac{1}{N} \sum_{i=1}^{\text{rank}\, g} s_i w_i$ and

(1) The sequence $(s_1, \ldots\ldots, s_n)$ is invariant under the permutation of indices corresponding to X *i.e.* $S_i = S_{\chi(i)}$, $1 \leq i \leq \mathrm{rank}\, g$.

(2) $N = \tau \sum_{i=0}^{O(g,\tau)} k_i^\tau s_i$, where k_i^τ are the Kac labels for $g^{(\tau)}$.

The order of this automorphism is N. We call δ a **shift vector**.

Proof : If α_i $i = 1, \ldots, \mathrm{rank}\, g$ are simple roots of g then (3.8) is equivalent to

$$\begin{aligned}
\sigma_{s;\tau}(E_{\alpha_j}) &= e^{\frac{2\pi i S_{[j]}}{N}} E_{X(\alpha_j)} \quad j = 1, \ldots, \mathrm{rank}\, g \\
&= e^{2\pi i \alpha_j . \delta} E_{X(\alpha_j)} \\
\sigma_{s;\tau}(E_{\Theta_0}) &= e^{\frac{2\pi i S_0}{N}} E_{X(\Theta_0)} \\
&= e^{2\pi i \Theta_0 . \delta} E_{X(\Theta_0)}
\end{aligned}$$

where δ is of the form given above. Now if we use the commutation relations to extend this to an automorphism of the whole algebra by induction then we have

$$\begin{aligned}
\sigma_{s;\tau}(E_{\alpha+\beta}) &= \frac{\sigma_{s;\tau}([E_\alpha, E_\beta])}{N_{\alpha,\beta}} \\
&= \frac{N_{X(\alpha),X(\beta)}}{N_{\alpha,\beta}} e^{2\pi i(\alpha+\beta).\delta} E_{X(\alpha)+X(\beta)} \\
&= e^{2\pi i(\alpha+\beta).\delta} E_{X(\alpha+\beta)}
\end{aligned}$$

using Lemma (3.1). This is also consistent with,

$$\begin{aligned}
\sigma_{s;\tau}([E_\alpha, E_{-\alpha}]) &= B_\alpha \sigma_{s;\tau}(\alpha.H) \\
&= B_\alpha X(\alpha).H \qquad \text{as } B_{X(\alpha)} = B_\alpha .
\end{aligned}$$

For the special case of inner automorphisms we have,

$$H \mapsto H, \tag{3.9}$$

$$E_\alpha \mapsto e^{2\pi i \alpha.\delta} E_\alpha, \tag{3.10}$$

where $\delta = \frac{1}{N} \sum_{i=1}^{\mathrm{rank}\, g} s_i w_i$ and $N = \sum_{i=0}^{\mathrm{rank}\, g} k_i s_i$, with the k_i just being the Kac labels on the EDD of g.

For a given automorphism of type (s;τ) let

$$\mathrm{g} = \bigoplus_{\mathrm{n} \in \mathbf{Z}_N} \mathrm{g_n}(\mathrm{s};\tau),$$

be the $\mathbb{Z}_N$ gradation associated with it.

Theorem (3.4) : Invariant subalgebras of finite order automorphisms :

(*i*) Let $i_1, \ldots\ldots, i_r$ be all the indices for which $\mathrm{s}_{i_1} = \ldots = \mathrm{s}_{i_r} = 0$. Then the invariant Lie subalgebra $\mathrm{g}_0(\mathrm{s};\tau)$ is isomorphic to the direct sum of the semi-simple Lie algebra obtained by removing all the vertices from $\mathrm{D}(\mathrm{g}^\tau)$, the Dynkin diagram of $\mathrm{g}^{(\tau)}$, corresponding to non-zero s_i, and an (n-r)-dimensional centre, $[\mathrm{u}(1)]^{\mathrm{n-r}}$.

(*ii*) Let $j_1, \ldots\ldots, j_s$ be all the indices for which $\mathrm{s}_{j_1} = \ldots = \mathrm{s}_{j_s} = 1$. Then the representation of $\mathrm{g}_0(\mathrm{s};\tau)$ on $\mathrm{g}_1(\mathrm{s};\tau)$ (or $\mathrm{g}_{-1}(\mathrm{s};\tau)$) is isomorphic to the direct sum of the s representations with highest weights -$\alpha_{j_1}, \ldots\ldots, -\alpha_{j_s}$ (or $\alpha_{j_1}, \ldots\ldots, \alpha_{j_s}$).

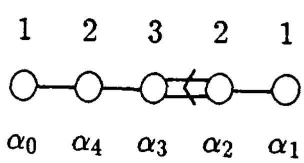
Proof : See [1] pp 97-98.

Recall that the **Dynkin indices** of a representation with highest weight Λ are given by,

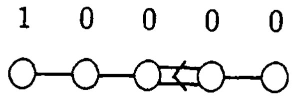
$$\mathrm{n}_i \equiv \frac{2\alpha_i.\Lambda}{\alpha_i^2}.$$

A few examples should help to elucidate these results.

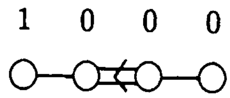
Examples : Let us look at some automorphisms of E_6 that are of the generic type (s; 2). The Dynkin diagram of $\mathrm{E}_6^{(2)}$ with Kac labels is;



1. Let $s_1 = (1,0,0,0,0)$ *i.e.*

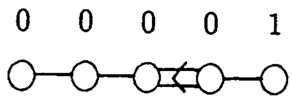


σ is of order $2.1=2$. $\mathfrak{g}_0(s_1;\,2) = F_4$ ($\dim \mathfrak{g}_0 = 52$). The Dynkin indices are

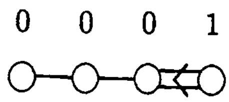


thus $\mathfrak{g}_1(s_1;\,2) = \underline{26}$.

2. Let $s_2 = (0,1,0,0,0)$ *i.e.*

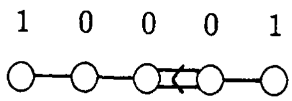


σ is of order $2.1=2$. $\mathfrak{g}_0(s_2;\,2) = C_4$ ($\dim \mathfrak{g}_0 = 36$). The Dynkin indices are

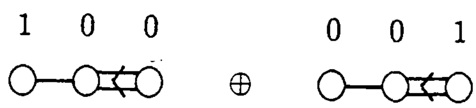


thus $\mathfrak{g}_1(s_2;\,2) = \underline{42}$.

3. Let $s_3 = (1,1,0,0,0)$ *i.e.*



σ is of order $2.2=4$. $\mathfrak{g}_0(s_3;\,2) = C_3 \oplus \mathbf{R}$ ($\dim \mathfrak{g}_0 = 22$). The Dynkin indices are



thus $\mathfrak{g}_1(s_3;\,2) = \mathfrak{g}_3(s_3;\,2) = \underline{6} \oplus \underline{14}$. Therefore $\dim \mathfrak{g}_2(s_3;\,2) = 16$.

Aside : One consequence of this classification of all the finite order automorphisms of $\mathfrak{g}$ is that it allows us to detemine all the real forms of a simple Lie algebra $\mathfrak{g}$ very easily. This is due to the fact that there is a real form associated with each non-conjugate involute (*i.e.* second order) automorphism of a complex semi-simple Lie algebra (see for eg Chapter 8 of [38]). The invariant subalgebra in this case corresponds to the maximal compact subalgebra of the real form.

Another (see also [38] pp 411-414) is that it allows us to determine all the maximal subalgebras of g of maximal rank *i.e.* $g' \subset g$ such that

(*i*) $\operatorname{rank} g' = \operatorname{rank} g$,

(*ii*) There is no g'' such that $g' \subset g'' \subset g$.

These are found from the extended Dynkin diagram of g by either knocking out

1. one spot whose Kac label is a prime number, or
2. one spot whose Kac label is one and the spot corresponding to α_0.

3.3 NON-CONJUGATE AUTOMORPHISMS WHICH FIX A CSA.

We now look at automorphisms of g which fix a given CSA, H. We denote the subgroups aut g and int g formed by such automorphisms as $\mathrm{aut}_H g$ and $\mathrm{int}_H g$ respectively. Given a Lie algebra g and a CSA H we can always choose a Cartan-Weyl basis. With respect to this basis an arbitrary automorphism, $\Sigma \in \mathrm{aut}_H g$, which maps the CSA into itself can always be written as the extension of an automorphism, $\sigma \in \mathrm{aut}\, \Phi_g$, of the root system as follows; $\Sigma : g \to g$ such that

$$\begin{aligned} \mathrm{H} &\mapsto \sigma(\mathrm{H}), \qquad \sigma \in \mathrm{aut}\,\Phi_g, \\ \mathrm{E}_\alpha &\mapsto \psi_\alpha \mathrm{E}_{\sigma(\alpha)}. \end{aligned} \tag{3.11}$$

Where,

$$\mathrm{N}_{\alpha,\beta} \mapsto \frac{\psi_\alpha \psi_\beta}{\psi_{\alpha+\beta}} \mathrm{N}_{\sigma(\alpha),\sigma(\beta)} \tag{3.12}$$

$$\psi_\alpha \psi_{-\alpha} = 1 \tag{3.13}$$

$$\psi_\alpha^* = \psi_{-\alpha} \tag{3.14}$$

(3.12) to (3.14) are necessary if Σ is to be an automorphism of g. From (3.13) and (3.14) we can see that,

$$\psi_\alpha \psi_\alpha^* = 1 \quad \text{therefore } \psi_\alpha \in \mathrm{S}^1 \ \forall\, \alpha \in \Phi_g.$$

Let us assume that the strucure constants are invariant under Σ so that,

$$\psi_\alpha \psi_\beta = \frac{\mathrm{N}_{\alpha,\beta}}{\mathrm{N}_{\sigma(\alpha),\sigma(\beta)}} \psi_{\alpha+\beta} \tag{3.15}$$

In the case of *simply laced algebras* we can replace $\mathrm{N}_{\alpha,\beta}$ by an appropriately chosen cocycle which is defined on the whole root lattice. Consequently we can extend all the sets of phases to be defined on the whole root lattice.

We can write $\Sigma = (\sigma, \psi)$ where $\sigma \in \mathrm{aut}\,\Phi_{\mathrm{g}}$ and $\psi : \Phi_{\mathrm{g}} \to \mathrm{S}^1$ is a projective representation of the root system with the factor set $\left\{\sigma(\alpha,\beta) \equiv \frac{\mathrm{N}_{\alpha,\beta}}{\mathrm{N}_{\sigma(\alpha),\sigma(\beta)}}\right\}$ which in addition satisfies (3.13) and (3.14). We shall call such a way of writing an automorphism a **twisted picture** of the automorphism. It is clear that $\mathrm{aut}_{\mathrm{H}}\mathrm{g}$ is isomorphic to the group of all projective representations of $\mathrm{aut}\,\Phi_{\mathrm{g}}$.

For a given automorphism σ of order n, $\sigma^n = 1$, let ω be a primitive n^{th} root of unity and $<\omega>$ the abelian group generated by ω. $<\omega>$ is isomorphic to $\mathbb{Z}_n$. We can write all the elements $\Sigma \in \mathrm{aut}_{\mathrm{H}}\mathrm{g}$ obtained by extending the automorphism $\sigma \in \mathrm{aut}\,\Phi_{\mathrm{g}}$ in the form

$$\begin{aligned} \mathrm{H} &\mapsto \sigma(\mathrm{H}), \\ \mathrm{E}_\alpha &\mapsto \psi^\sigma_\alpha \Delta_\alpha \mathrm{E}_{\sigma(\alpha)}. \end{aligned}$$

Here $\psi^\sigma : \Phi_{\mathrm{g}} \to <\omega>$ is a *particular* choice of projective representation and $\Delta : \Phi_{\mathrm{g}} \to \mathbb{C} - \{0\}$ is an arbitrary homomorphism *i.e.*

$$\Delta_\alpha \Delta_\beta = \Delta_{\alpha+\beta} \qquad \forall\, \alpha, \beta \in \Phi_{\mathrm{g}}. \tag{3.16}$$

Thus the phases $\{\Delta_\alpha \mid \alpha \in \Phi_{\mathrm{g}}\}$ form an ordinary representation of Φ_{g}. The corresponding *trivial automorphisms* or changes of basis are obtained by extending the identity automorphism, $\mathbb{1} : \Phi_{\mathrm{g}} \to \Phi_{\mathrm{g}}$, to the whole algebra,

$$\begin{aligned} \mathrm{H} &\mapsto \mathrm{H}, \\ \mathrm{E}_\alpha &\mapsto \Delta_\alpha \mathrm{E}_\alpha. \end{aligned}$$

We shall call this extension $\mathbb{1}_\Delta$. The set of all such automorphisms, which we can look on as **gauge transformations** for the basis of g, form a subgroup of $\mathrm{aut}_{\mathrm{H}}\mathrm{g}$ which we shall call Π_{g}. Π_{g} is isomorphic to the set of maps $\Delta(\underline{\omega}) : \Phi_{\mathrm{g}} \to \mathbb{C} - \{0\}$ parametrised by $\underline{\omega} \in (\mathbb{C} - \{0\})^{\mathrm{rank}\,\mathrm{g}}$ which corresponds to a particular choice of phases on the simple roots. A given Δ is obtained by defining $\Delta_{\alpha_i} = \omega_i$ and extending the definition of Δ to the whole lattice by use of the homomorphism (3.16).

In brief, an arbitrary extension of $\sigma \in \mathrm{aut}\,\Phi_{\mathrm{g}}$ to an element $\Sigma \in \mathrm{aut}_{\mathrm{H}}\mathrm{g}$ can be written in the form $\Sigma = \Sigma^\sigma \mathbb{1}_\Delta$, where $\mathbb{1}_\Delta \in \Pi_{\mathrm{g}}$ and Σ^σ is the particular extension with phases $\{\psi^\sigma_\alpha\}$.

Theorem (3.5) :

$$\text{(1)} \quad \frac{\mathrm{aut_H\,g}}{\Pi_{\mathrm{g}}} \cong \mathrm{aut}\;\Phi_{\mathrm{g}}$$

$$\text{(2)} \quad \mathrm{aut_H\,g} \cong \Pi_{\mathrm{g}} \rtimes \mathrm{aut}\;\Phi_{\mathrm{g}}$$

Proof :

(1) Let $\Theta : \mathrm{aut_H g} \rightarrow \mathrm{aut}\;\Phi_{\mathrm{g}}$ such that $(\sigma, \psi) \mapsto \sigma$. Almost trivially Θ is a homomorphism as

$$\begin{aligned} \Theta((\sigma,\psi)\circ(\tau,\phi)) &= \Theta((\sigma\tau,\zeta)) \\ &= \sigma\tau \end{aligned}$$

where $\zeta_\alpha = \phi_\alpha \psi_{\tau(\alpha)}\ \forall\ \alpha \in \Phi_{\mathrm{g}}$.

Whereas $\Theta((\sigma,\psi))\Theta((\tau,\phi)) = \sigma\tau$

As $\mathrm{Ker}\,\Theta = \Pi_{\mathrm{g}}$ and $\mathrm{Im}\,\Theta = \mathrm{aut}\;\Phi_{\mathrm{g}}$ we have $\Pi_{\mathrm{g}} \triangleleft \mathrm{aut_H g}$ and thus (1) is true by the First Isomorphism Theorem.

(2) By (1) $\mathrm{aut_H g} \cong \Pi_{\mathrm{g}} \rtimes \mathrm{aut}\;\Phi_{\mathrm{g}}$ where the semi-direct product multiplication is given by

$$(\sigma,\psi)\circ(\tau,\phi) = (\sigma\tau,\zeta) \ \text{ where } \zeta_\alpha = \phi_\alpha\psi_{\tau(\alpha)}\ \forall\ \alpha \in \Phi_{\mathrm{g}}.$$

We shall define,

$$\mathrm{aut}_{\mathrm{H}}^{[\pi]}\mathrm{g} \equiv \frac{\mathrm{aut_H\,g}}{\Pi_{\mathrm{g}}}.$$

Notice that due to the isomorphism $\mathrm{aut}_{\mathrm{H}}^{[\pi]}\mathrm{g} \cong \mathrm{aut}\;\Phi_{\mathrm{g}}$, all the elements of $\mathrm{aut}_{\mathrm{H}}^{[\pi]}\mathrm{g}$ must be of finite order and thus can be written in the form (3.8). We shall call this way of writing the automorphism the **shifted picture**.

Similiarly by replacing $\sigma \in \mathrm{aut}\;\Phi_{\mathrm{g}}$ by $\sigma \in \mathrm{W_g}$ in the above we have,

$$\mathrm{int}_{\mathrm{H}}^{[\pi]}\mathrm{g} \equiv \frac{\mathrm{int_H\,g}}{\Pi_{\mathrm{g}}}$$

and

$$\mathrm{int}_{\mathrm{H}}^{[\pi]}\mathrm{g} \cong \mathrm{int}\;\Phi_{\mathrm{g}} \cong \mathrm{W_g},$$

with

$$\mathrm{int}_{\mathrm{H}}\,\mathrm{g} \cong \Pi_{\mathrm{g}} \rtimes \mathrm{W}_{\mathrm{g}}.$$

It shall turn out that non-conjugate elements of $\mathrm{aut}_{\mathrm{H}}^{[\pi]}\,\mathrm{g}$ lead to non-equivalent gradations of the Kac-Moody algebras. Thus our interest in gradations of Kac-Moody algebras naturally leads us to study the conjugacy classes of W_{g} and $\mathrm{aut}\,\Phi_{\mathrm{g}}$ which we do in Chapters 4 and 5. We are also interested in picking a particular representative Σ^{σ} for each conjugacy class of $\mathrm{aut}_{\mathrm{H}}^{[\pi]}\,\mathrm{g}$. We will demonstrate the existence of a choice of phases which will turn out to be very useful in later calculations.

Lemma (3.6) : For simply laced algebras we can always choose the phases $\{\psi_{\alpha}^{\sigma}\}$ such that $\psi_{\alpha}^{\sigma} = 1$ if $\sigma(\alpha) = \alpha$, which means that,

$$\sigma(\alpha) = \alpha \;\Leftrightarrow\; \Sigma^{\sigma}(\mathrm{E}_{\alpha}) = \mathrm{E}_{\alpha}.$$

Proof : Let ϕ_{α}^{σ} be a particular choice of phases and define

$$\Lambda_{\mathrm{R}}^{0} = \{\alpha \in \Lambda_{\mathrm{R}} \mid \sigma(\alpha) = \alpha\} \subset \Lambda_{\mathrm{R}},$$
$$\Phi_{\mathrm{g}}^{0} = \{\alpha \in \Phi_{\mathrm{g}} \mid \sigma(\alpha) = \alpha\} \subset \Phi_{\mathrm{g}},$$

to be the invariant root lattice and root system respectively. Now on Φ_{g}^{0} (3.15) with $\mathrm{N}_{\alpha,\beta}$ replaced by a suitable cocycle $\varepsilon(\alpha,\beta)$ reduces to,

$$\phi_{\alpha}^{\sigma}\phi_{\beta}^{\sigma} = \phi_{\alpha+\beta}^{\sigma} \qquad \forall\, \alpha,\beta \in \Phi_{\mathrm{g}}^{0}.$$

Hence these phases form a representation of the lattice $\Phi_{\mathrm{g}}^{0} \subset \Phi_{\mathrm{g}}$. This representation can be extended to form a representation, $\{\psi_{\alpha}\}$, of the whole root lattice such that $\psi_{\alpha} = \phi_{\alpha}^{\sigma}$ on Φ_{g}^{0} and

$$\psi_{\alpha}\psi_{\beta} = \psi_{\alpha+\beta} \qquad \forall\, \alpha,\beta \in \Phi_{\mathrm{g}}. \tag{3.17}$$

We do this by choosing a basis α_i of Φ_{g}^{0} and extending it to a basis for the whole root system, Φ_{g}, by picking $\beta_i \in \Phi_{\mathrm{g}} - \Phi_{\mathrm{g}}^{0}$. Now define $\psi_{\alpha_i} = \phi_{\alpha_i}^{\sigma}$ and $\psi_{\beta_i} = 1$ and extend them to $\psi : \Phi_{\mathrm{g}} :\rightarrow \mathrm{S}^1$ by using (3.17). The required phases are then obtained by defining,

$$\psi_{\alpha}^{\sigma} \equiv \frac{\phi_{\alpha}^{\sigma}}{\psi_{\alpha}} \qquad \forall\, \alpha \in \Phi_{\mathrm{g}}.$$

3.4 ARBITRARY INTEGER GRADED REALIZATIONS OF THE KAC-MOODY ALGEBRAS.

Let g be a simple finite dimensional Lie algebra and let $g^{(\tau)}$ $\tau = 1, 2$ or 3 be the corresponding Kac-Moody algebras. Let $\{E_\alpha, h_i \mid \alpha \in \Phi_g,\ i = 1, ..., \mathrm{rank}\, g\}$ be a Cartan-Weyl basis for g.

We shall realize all the infinite dimensional algebras as central extensions of infinite dimensional **loop algebras**,

$$L(g) = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} g,$$

and subalgebras of them. $\mathbb{C}[t, t^{-1}]$ denotes the **algebra of Laurent polynomials** in t,

$$\mathbb{C}[t, t^{-1}] = \{a(t) = \sum_{n \in Z} a_n t^n \mid \text{all but a finite no. of } a_n = 0\}.$$

The commutation relation on L(g) is defined as

$$[a(t) \otimes x, b(t) \otimes y] = a(t)b(t) \otimes [x, y].$$

Hence L(g) is just the Lie algebra of regular rational maps $f : \mathbb{C} - \{0\} \to g$ such that $z \mapsto \sum_{n \in Z} x_n z^n$ where $x_n \in g$. For $\mid z \mid = 1$ we have a **loop** in $\mathbb{C} - \{0\}$ so that f describes a mapping from S^1 to g, hence the name loop algebras.

We extend L(g) to $L(g) \oplus \mathbb{C}C$ by adding a one-dimensional centre C, and extending the commutation relations so that

$$[a(t) \otimes x + \lambda C, b(t) \otimes y + \mu C] = a(t)b(t) \otimes [x, y] + \mathrm{Res}\left(\frac{da}{dt} b\right) B(x, y)C. \qquad (3.18)$$

where $\mu, \lambda \in \mathbb{C}$ and

(*i*) Res:$\mathbb{C}[t, t^{-1}] \to \mathbb{C}$ such that $\mathrm{Res}(a(t)) = a_{-1}$, the normal residue of the Laurent polynomial.

(*ii*) B:$g \times g \to \mathbb{C}$ such that $B(x, y) = \mathrm{Tr}\{\mathrm{ad}x\, \mathrm{ad}y\}$ is the Killing form on g.

Notice that $\Psi(a(t) \otimes x, b(t) \otimes y) \equiv \mathrm{Res}(\frac{da}{dt} b)B(x, y)$ is a bilinear complex valued function satisfying

1. $\Psi(a,b) = -\Psi(b,a)$

2. $\Psi(a,[b,c]) + \Psi(b,[c,a]) + \Psi(c,[a,b]) = 0$ a, b, c $\in L(g)$ and hence Ψ is a complex valued 2-cocycle on $L(g)$. This is required if the commutation relations on $L(g)$ are to satisfy the Jacobi Identity.

The natural gradation, d, of $L(g)\oplus\mathbb{C}C$ acts like $t\frac{d}{dt}$ on $L(g)$ and kills C.

3.4.1. A homogeneous gradation of $g^{(1)}$.

The centrally extended loop algebra, $\hat{L} \equiv L(g) \oplus \mathbb{C}C$, is isomorphic to $g^{(1)}$, see for example [1].

If we now write $E^n_\alpha \equiv t^n \otimes E_\alpha$, $h^n_i \equiv t^n \otimes h_i$ then in terms of this basis the commutation relations of the untwisted Kac-Moody algebra become,

$$[h^m_i, h^n_j] = mB(h_i,h_j)C\delta_{m+n,0}, \tag{3.19}$$

$$[\alpha.H^m, E^n_\beta] = \alpha.\beta E^{m+n}_\beta, \tag{3.20}$$

$$[E^m_\alpha, E^n_\beta] = \begin{cases} N_{\alpha,\beta}E^{m+n}_{\alpha+\beta} & \alpha+\beta \in \Phi_g, \quad (3.21) \\ B_\alpha(\alpha.H^{m+n} + mC)\delta_{m+n,0} & \alpha+\beta = 0, \quad (3.22) \\ 0 & \alpha+\beta \notin \Phi_g, \quad (3.23) \end{cases}$$

$$[C, g^{(1)}] = 0 \tag{3.24}$$

where

1.
$$\alpha.H^m = n_i h^m_i \quad \text{if} \quad \alpha = \sum_{i=1}^{\mathrm{rank}\, g} n_i\alpha_i.$$

2. The gradation, d, is given by

$$[d, E^m_\alpha] = mE^m_\alpha, \quad [d, h^m_i] = mh_i, \quad [d, C] = 0.$$

In addition we choose $B(h_i,h_j) = \delta_{ij}$ and the B_α so that we have a Chevalley basis of g and hence of $g^{(1)}$. This realization corresponds to a homogeneous gradation whose horizontal algebra is g. A suitable set of *Chevalley generators* for $g^{(1)}$ is given by $\{E^1_{\alpha_i}, E^1_{-\Theta} \mid i = 1, ..., \mathrm{rank}\, g\}$ where $\Theta \in \Phi_g$ is the highest root of g. With respect to this basis the *simple roots* are $\{\alpha_0 \equiv \delta - \Theta,\ \alpha_1, \ldots\ldots, \alpha_n\}$ whilst the corresponding *simple co-roots* are $\{B_\Theta C - \Theta.H^0, \alpha_i.H^0, \ldots\ldots, \alpha_n.H^0\}$. The *root lattice* is given by,

$$\Phi_{g^{(1)}} = \{\alpha + m\delta,\ \alpha \in \Phi_g,\ m \in \mathbb{Z}\} \cup \{m\delta,\ m \in \mathbb{Z} - \{0\}\}.$$

The roots $\alpha + m\delta$ have positive norm, are non-degenerate and are called **real roots.** The roots $m\delta$ have zero norm, are $n = \operatorname{rank} g$ degenerate and are called **imaginary** or **null roots.**

3.4.2. Arbitrary gradations of $g^{(\tau)}$.

Let $\sigma \in \operatorname{aut} \Phi_g$ be a root system automorphism of order n, and $\Sigma = (\sigma, \psi) \in \operatorname{aut}_H g$, of finite order N, an extension of σ to a Lie algebra automorphism. We can in turn extend this to a Kac-Moody algebra automorphism, $\hat{\Sigma}$, where,

$$\begin{aligned} \hat{\Sigma}(t^m \otimes x) &= \omega^{-m} t^m \otimes \Sigma(x) \quad m \in \mathbb{Z},\ x \in g, \\ \hat{\Sigma}(C) &= C, \end{aligned}$$

and ω is an N^{th} root of unity. Let

$$\hat{L}(g, \Sigma) \equiv \bigoplus_{m \in Z} (t^m \otimes g_{m\,(\mathrm{mod}\,N)}) \oplus \mathbb{C}C \tag{3.25}$$

be the invariant subalgebra of $\hat{L}(g)$ under $\hat{\Sigma}$, where

$$g = \bigoplus_{m \in Z_N} g_m$$

is the eigenspace decomposition of g under Σ.

$\hat{L}(g, \Sigma)$ is the Kac-Moody algebra of **equivariant maps** from $\mathbb{C} - \{0\}$ to g that are invariant under the automorphism,

$$\begin{aligned} f &\mapsto \Sigma \circ f \circ \omega^{-1}, \\ \textit{i.e.}\quad \Sigma f(c) &= f(\omega c) \qquad \forall\, c \in \mathbb{C} - \{0\}. \end{aligned}$$

[c.f. twisted string boundary conditions $\sigma X(z) = X(e^{2\pi i} z)$.] Now if $\Sigma \in \Pi_g$ is a gauge transformation on g then $\hat{\Sigma}$ is just a gauge transformation on $g^{(1)}$. As a result any two

Lie algebra automorphisms differing by an element of Π_g give the same gradation. If σ and σ' are conjugate in aut Φ_g, $\sigma' = x^{-1}\sigma x$, then let us define

$$s_\alpha \equiv \phi^{-1}_{\sigma x(\alpha)}\psi_{x(\alpha)}\phi_\alpha,$$

where $X = (x, \phi)$ is the corresponding Lie algebra automorphism. We have

$$\begin{aligned} s_\alpha s_\beta &= \phi^{-1}_{\sigma x(\alpha)}\phi^{-1}_{\sigma x(\beta)}\psi_{x(\alpha)}\psi_{x(\beta)}\phi_\alpha\phi_\beta, \\ &= \frac{N_{\sigma x(\alpha),\sigma x(\beta)}}{N_{x^{-1}\sigma x(\alpha),x^{-1}\sigma x(\beta)}}\frac{N_{x(\alpha),x(\beta)}}{N_{\sigma x(\alpha),\sigma x(\beta)}}\frac{N_{\alpha,\beta}}{N_{x(\alpha),x(\beta)}}s_{\alpha+\beta}, \\ &= \frac{N_{\alpha,\beta}}{N_{\sigma'(\alpha),\sigma'(\beta)}}s_{\alpha+\beta}. \end{aligned}$$

Thus $\psi'_\alpha = \Delta_\alpha s_\alpha$ for some set of phases Δ_α satisfying (3.16) and therefore $X^{-1}\Sigma X = \Sigma' \mathbb{1}_\Delta$, where $\mathbb{1}_\Delta \in \Pi_g$ and $\Sigma' = (\sigma', \psi')$. Consequently $X^{-1}\Sigma X$ and Σ' give equivalent gradatitions. Thus conjugate automorphisms $\sigma \in$ aut Φ_g lead to equivalent gradations. in fact non-conjugate automorphisms lead to different gradations so that the gradations are in one–to–one correspondence with the conjugacy classes of $\mathrm{aut}_H^{[\tau]}\, g$ which are in turn in one–to–one correspondence with the conjugacy classes of aut Φ_g [16].

If we choose Σ to be a diagram automorphism, X, of g of order τ, or 2τ if $g = A_{2n}$, then we get a homogeneous gradation of the corresponding Kac-Moody algebra $g^{(\tau)}$ [1].

$$g^{(\tau)} \cong \hat{L}(g, X) = \begin{cases} \bigoplus\limits_{m \in Z} t^m \otimes g_{m\,(\mathrm{mod}\,\tau)} \oplus \mathbb{C}C & g \neq A_{2n} \\ \bigoplus\limits_{m \in Z} t^m \otimes g_{m\,(\mathrm{mod}\,2\tau)} \oplus \mathbb{C}C & g = A_{2n} \end{cases}$$

Notice in particular that $\hat{L}(g, \mathbb{1}) \cong \hat{L}(g) \cong g^{(1)}$.

All algebra automorphisms consist of a product of a diagram automorphism (maybe the identity) and an inner automorphism, $\Sigma = X\Sigma_{\mathrm{INNNER}}$. As all inner automorphisms can be rewritten in terms of a shift vector (3.9) and (3.10) it can be seen that although they will change the gradation they will not alter the underlying Kac-Moody algebra *i.e.*

$$\hat{L}(g, \Sigma) \cong \hat{L}(g, X) \cong g^{(\tau)}.$$

So conjugacy classes of outer automorphisms in aut Φ_g lead to different gradations of the twisted Kac–Moody algebra (or algebras in the case of D_4) whilst the conjugacy classes of inner automorphisms give different gradations of the untwisted Kac–Moody algebra.

To see an example of this change of gradation let us look at the case of an inner automorphism on $\hat{\mathrm{L}}(\mathrm{g})$. Here we have

$$\begin{aligned} \hat{\Sigma}(\mathrm{t}^{\mathrm{m}} \otimes \mathrm{E}_\alpha) &= \omega^{-\mathrm{m}} e^{2\pi i \alpha.\delta} \mathrm{t}^{\mathrm{m}} \otimes \mathrm{E}_\alpha, \\ &= \omega^{-\mathrm{m}+\mathrm{N}\alpha.\delta} \mathrm{t}^{\mathrm{m}} \otimes \mathrm{E}_\alpha, \\ \hat{\Sigma}(\mathrm{t}^{\mathrm{m}} \otimes \mathrm{h}_i) &= \mathrm{t}^{\mathrm{m}} \otimes \mathrm{h}_i, \\ \hat{\Sigma}(\mathrm{C}) &= \mathrm{C}. \end{aligned}$$

Thus the gradation $\hat{\mathrm{L}}(\mathrm{g}, \Sigma)$ has

$$\{\mathrm{t}^{\mathrm{m}+\mathrm{N}\alpha.\delta} \otimes \mathrm{E}_\alpha,\ \mathrm{t}^{\mathrm{m}} \otimes \mathrm{h}_i,\ \mathrm{C} \mid \mathrm{m} \in \mathbb{Z},\ i = 1, ..., \operatorname{rank} \mathrm{g}\} \tag{3.26}$$

as a basis. We will also sometimes write the gradation of $\mathrm{g}^{(\tau)}$ produced by the extension of $\sigma \in \operatorname{aut} \Phi_{\mathrm{g}}$ to an algebra automorphism as $\mathrm{g}^{(\tau)}(\sigma)$.

The *homogeneous gradation* corresponds to taking $\sigma = 1\!\!1$, whilst the *principal gradation* is given by taking $\sigma = \mathrm{w}_1.....\mathrm{w}_{\operatorname{rank} \mathrm{g}}$, the **Coxeter element**. Here w_i denotes a Weyl reflection in the simple root α_i.

Commutation relations :

Let $\mathrm{P}_\mathrm{n} : \mathrm{g} \rightarrow \mathrm{g}_\mathrm{n}$ be the projection of g onto the eigenspace g_n. If $\Sigma = (\sigma, \psi)$ is a Lie algebra automorphism of order N satisfying (3.11) to (3.14) then P_n is given by,

$$\mathrm{P}_\mathrm{n} = \frac{1}{\mathrm{N}} \sum_{r=0}^{\mathrm{N}-1} \omega^{-\mathrm{n}r} \Sigma^r .$$

Hence defining $\eta(r, \alpha) = \psi_\alpha.....\psi_{\sigma^r(\alpha)}$ so that

$$\Sigma^r(\mathrm{E}_\alpha) = \eta(r, \alpha) \mathrm{E}_{\sigma^r(\alpha)}$$

we have,

$$\begin{aligned} \mathrm{P}_\mathrm{n}(\mathrm{E}_\alpha) &= \frac{1}{\mathrm{N}} \sum_{r=0}^{\mathrm{N}-1} \omega^{-\mathrm{n}r} \eta(r, \alpha) \mathrm{E}_{\sigma^r(\alpha)} , \\ \mathrm{P}_\mathrm{n}(\alpha.\mathrm{H}) &= \frac{1}{\mathrm{N}} \sum_{r=0}^{\mathrm{N}-1} \omega^{-\mathrm{n}r} \sigma^r(\alpha).\mathrm{H} . \end{aligned}$$

We now introduce the following basis for the Kac-Moody algebra gradation $\hat{\mathrm{L}}(\mathrm{g}, \Sigma)$;

$$\begin{aligned}
\mathrm{E}_\alpha(\mathrm{n}) &\equiv \mathrm{t}^{\mathrm{n}} \otimes \mathrm{P}_{\mathrm{n}}(\mathrm{E}_\alpha)\,, \\
&= \mathrm{t}^{\mathrm{n}} \otimes \frac{1}{\mathrm{N}} \sum_{r=0}^{\mathrm{N}-1} \omega^{-\mathrm{n}r} \eta(r,\alpha) \mathrm{E}_{\sigma^r(\alpha)}\,, \\
\alpha(\mathrm{n}) &\equiv \mathrm{t}^{\mathrm{n}} \otimes \mathrm{P}_{\mathrm{n}}(\alpha.\mathrm{H})\,, \\
&= \mathrm{t}^{\mathrm{n}} \otimes \frac{1}{\mathrm{N}} \sum_{r=0}^{\mathrm{N}-1} \omega^{-\mathrm{n}r} \sigma^r(\alpha).\mathrm{H}\,, \\
\mathrm{C}.
\end{aligned}$$

Using (3.18) we can show that the commutation relations in terms of this basis are

$$\begin{aligned}
[\alpha(\mathrm{m}), \beta(\mathrm{n})] &= \mathrm{m}\, \delta_{\mathrm{m}+\mathrm{n},0} \mathrm{P}_{\mathrm{m}}(\alpha).\beta\,, \\
&= \frac{\mathrm{m}}{\mathrm{N}} \delta_{\mathrm{m}+\mathrm{n},0} \sum_{s=0}^{\mathrm{N}-1} \omega^{-\mathrm{m}s} \sigma^s(\alpha).\beta\,, \qquad (3.27) \\
[\alpha(\mathrm{m}), \mathrm{E}_\beta(\mathrm{n})] &= \mathrm{t}^{\mathrm{n}} \otimes \mathrm{P}_{\mathrm{m}}(\alpha).\frac{1}{\mathrm{N}} \sum_{s=0}^{\mathrm{N}-1} \omega^{-\mathrm{m}s} \sigma^s(\beta) \eta(s,\beta) \mathrm{E}_{\sigma^s(\beta)}\,, \qquad (3.28) \\
[\mathrm{E}_\alpha(\mathrm{m}), \mathrm{E}_\beta(\mathrm{n})] &= \frac{1}{\mathrm{N}} \left\{ \sum_{\substack{x=0 \\ \sigma^x(\alpha)+\beta \in \Phi_{\mathrm{g}}}}^{\mathrm{N}-1} \omega^{-\mathrm{m}x} \eta(x,\alpha) \mathrm{N}_{\sigma^x(\alpha),\beta} \mathrm{E}_{\sigma^x(\alpha)+\beta}(\mathrm{m}+\mathrm{n}) \right. \\
&\quad \left. + \sum_{\substack{x=0 \\ \sigma^x(\alpha)+\beta=0}}^{\mathrm{N}-1} \omega^{-\mathrm{m}x} \eta(x,\alpha) \mathrm{B}_{-\beta}(\mathrm{m}\ \delta_{\mathrm{m}+\mathrm{n},0}\mathrm{C} - \beta(\mathrm{m}+\mathrm{n})) \right\}. \qquad (3.29)
\end{aligned}$$

Notice that for simply-laced algebras

$$\begin{aligned}
\sigma^x(\alpha) + \beta \in \Phi_{\mathrm{g}} &\Leftrightarrow \sigma^x(\alpha).\beta = -1, \\
\sigma^x(\alpha) + \beta = 0 &\Leftrightarrow \sigma^x(\alpha).\beta = -2,
\end{aligned}$$

and $\mathrm{N}_{\sigma^x(\alpha),\beta}$ can be replaced by $\varepsilon(\sigma^x(\alpha),\beta)$ where ε is a suitable 2-cocycle.

In particular when we look at the commutation relations of the invariant subalgebra g_0 we find

$$\begin{aligned}
[\alpha(0), \beta(0)] &= 0, \qquad (3.30) \\
[\alpha(0), \mathrm{E}_\beta(0)] &= \alpha_0.\beta \mathrm{E}_\beta(0), \qquad (3.31)
\end{aligned}$$

$$[\mathrm{E}_\alpha(0), \mathrm{E}_\beta(0)] = \frac{1}{\mathrm{N}}\left\{ \sum_{\substack{x=0 \\ \sigma^x(\alpha)+\beta \in \Phi_g}}^{\mathrm{N}-1} \eta(x,\alpha) \mathrm{N}_{\sigma^x(\alpha),\beta} \mathrm{E}_{\sigma^x(\alpha)+\beta}(0) \right.$$
$$\left. - \sum_{\substack{x=0 \\ \sigma^x(\alpha)+\beta = 0}}^{\mathrm{N}-1} \eta(x,\alpha) \mathrm{B}_{-\beta}\beta(0) \right\}. \tag{3.32}$$

where $\tilde{\alpha} \equiv \alpha + \sigma(\alpha) + ... + \sigma^{\mathrm{N}-1}(\alpha)$, and $\alpha_0 \equiv \frac{\tilde{\alpha}}{\mathrm{N}}$.

4. Automorphisms of the Root System.

The symmetry group of the root system of a simple Lie algebra g, aut Φ_g, is the semi-direct product of the Weyl group, W_g, and the factor group of Dynkin diagram symmetries, Γ_g,

$$\text{aut}\,\Phi_g \;=\; W_g \rtimes \Gamma_g. \tag{4.1}$$

We can write an element of aut Φ_g as (σ,Y) where $\sigma \in W_g$, $Y \in \Gamma_g$. (σ,Y) acts on on α by sending it to $\sigma(Y(\alpha))$.

<u>Semi – direct product multiplication :</u> $(\sigma, Y).(\tau, Z) = (\sigma Y \tau Y^{-1}, YZ)$

The group aut Φ_g permutes the set of roots and so must be isomorphic to a subgroup of the symmetric group (i.e. group of permutations) on Φ_g. In particular it must be finite. Information about the automorphism groups is given in Table 4.1. [39].

<u>Table 4.1 :</u> *Information about the automorphism groups of the simple Lie algebra root systems.*

g	$\lvert\Phi_g{}^+\rvert$	W_g	Γ_g	$\lvert W_g\rvert$
A_n	$\binom{n+1}{2}$	S_{n+1}	$\mathbb{Z}_2$ $(n\geq 2)$	$(n+1)!$
B_n,C_n	n^2	$(\mathbb{Z}_2)^n \rtimes S_n$ †	1	$2^n n!$
D_n	$n(n-1)$	$(\mathbb{Z}_2)^{n-1} \rtimes S_n$	S_3 $n=4$ $\mathbb{Z}_2$ $n>4$	$2^{n-1}n!$
E_6	36		$\mathbb{Z}_2$	$2^7.3^4.5 = 51,840$
E_7	63		1	$2^{10}.3^4.5.7 = 2,903,040$
E_8	120		1	$2^{14}.3^5.5^2.7 = 696,729,600$
F_4	24		1	$2^7.3^2 = 1152$
G_2	6	D_6	1	$2^2.3 = 12$

† W_{B_n} and W_{C_n} are both isomorphic to the **hyperoctahedral group** of order $2^n n!$.
D_6 = **dihedral group** of order 12.

In this chapter we shall elucidate the conjugacy classes of both the Weyl group and the full automorphism group.

4.1 Conjugacy classes of the Weyl group, W_g.

The conjugacy classes of the Weyl groups were initially determined individually by a number of authors. However here we shall describe a more unified approach due to Carter [3].

The **Weyl group** of a root lattice, Λ_R, is the group generated by reflections in the hyperplanes orthogonal to roots in the root system Φ_g *i.e.* $w_\beta : \Lambda_R \mapsto \Lambda_R$ such that

$$w_\beta(\alpha) = \alpha - 2\frac{\alpha.\beta}{\beta^2}\beta \quad \alpha \in \Lambda_R, \beta \in \Phi_g \subset \Lambda_R. \tag{4.2}$$

(From now on we will write 'reflection in a root α' when we really mean 'reflection in the hyperplane orthogonal to the root α'.)

The set of hyperplanes perpendicular to all the roots in the root system partition V into disjoint cones called **Weyl chambers**. The number of Weyl chambers is equal to the order of the Weyl group. There is one Weyl chamber, $C(\Delta)$, such that for any $\beta \in C(\Delta)$ $\alpha_i.\beta \geq 0$ for all the simple roots. It is called the **fundamental Weyl chamber**. A vector β lying in or along the walls of $C(\Delta)$ is said to be **dominant** (*i.e.* $\alpha_i.\beta \geq 0$ $i = 1, ..., \mathrm{rank}\, g$), whilst a vector lying completely inside $C(\Delta)$ is said to be **strongly dominant** (*i.e.* $\alpha_i.\beta > 0$ $i = 1, ..., \mathrm{rank}\, g$).

It can be shown (see for e.g. [3]) that any element, $w \in W_g$, can be written as a (not necessarily unique) product of rank g or less weyl reflections in linearly independent roots, $\beta_i \in \Lambda_R$,

$$w = w_1 w_2 .....w_k \qquad 1 \leq k \leq \mathrm{rank}\, g. \tag{4.3}$$

where w_i denotes a weyl reflection in the root β_i. Let $l(w)$ denote the smallest possible value of k.

Lemma (4.1) : $l(w)$ is the number of eigenvalues of w on V which are not equal to 1.

Proof : See [3].

For a given set of , not necessarily simple, roots $\beta_i \in \Lambda_R$ $i = 1, ...,$l(w) we can draw a graph which contains all the relevant information about the inner products between the roots; just like the usual Dynkin diagram. A **graph** is a set of of l(w) nodes, one for each root, such that any two nodes are joined by $4\cos^2\Theta$ links, where Θ is the angle between the roots corresponding to the nodes. In addition if α and β are not perpendicular then an arrow is drawn on the links, pointing from node α to node β if $\alpha^2 > \beta^2$.

A **subgraph** of a graph is a subset of nodes, together with the bond joining them to each other. A **cycle** is a graph in which each node is joined to just two others by bonds with one or more links (see Fig. 4.1).

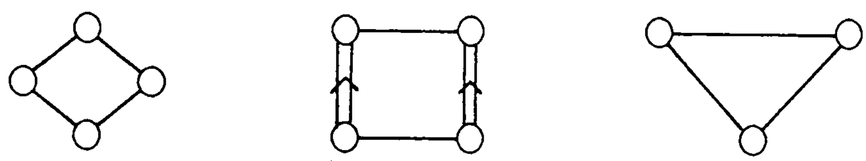


Figure 4.1 : Examples of cycles.

A graph, Γ, is called an **admissible diagram** if

(1) The nodes of Γ correspond to a set of linearly independent roots.

(2) Each subgraph of Γ which is a cycle contains an even number of nodes.

So the first two cycles in figure 4.1 correspond to admissible diagrams, assuming the roots are linearly independent, whilst the third one does not.

The order in which the reflections in roots, corresponding to nodes of the graph, are performed is important as different orderings can lead to the resulting element being in different conjugacy classes. One of the main results of [3] is to show that every element of a Weyl group is expressible as the product of two involutions. Thus any weyl reflection can be written as the product of weyl reflections performed in one set of mutually orthogonal roots followed by the product of weyl reflections in a second set of mutually orthogonal roots,

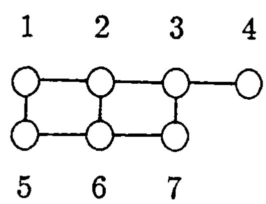
$$\mathrm{w} = \mathrm{w}_{\beta_1}...\mathrm{w}_{\beta_k}\mathrm{w}_{\gamma_1}...\mathrm{w}_{\gamma_l} \quad 1 \leq k+l \leq \mathrm{rank}\,\mathrm{g}, \tag{4.4}$$

such that:

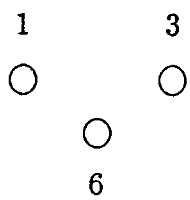
$$\beta_i.\beta_j = 0 = \gamma_i.\gamma_j \qquad \text{if } i \neq j. \tag{4.5}$$

The convention for the ordering of the reflections is to split the roots of a diagram into two sets of mutually orthogonal roots (which by the above must always be possible) and then perform the reflections in one set followed by the reflections in the other.

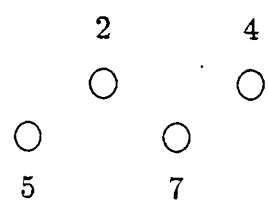
E.g.



can be separated into



and



and therefore would represent an element $w_1 w_3 w_6 w_2 w_4 w_5 w_7$.

We shall now state some of the main results of [3] refering the reader to the original reference for further details and proofs.

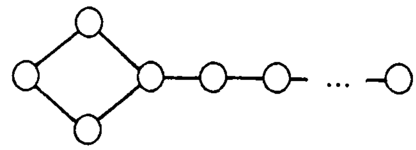
Admissible diagrams :

(1) All admissible diagrams for a given Weyl group, W_g, which *do not* contain cycles correspond to a Dynkin diagram of some Weyl subgroup of W_g. These can all be found [40,41] by removing one or more nodes in all possible ways from the extended Dynkin diagram of Φ_g. The resulting diagrams and their duals (obtained by interchanging long and short roots) should then be taken and the process repeated any number of times.

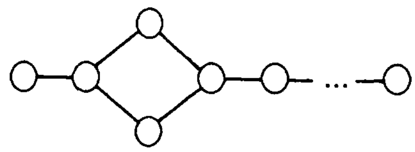
(2) All admissible diagrams for W_g which *do* contain cycles can be obtained from the ones which do not contain them. To do this one takes all the diagrams without cycles and wherever possible replaces any connected component Γ_i of the graph by any corresponding admissible diagram with cycles associated with the Weyl group of Γ_i but with no proper subgroup of it. All such diagrams are listed in Table 4.2.

Table 4.2 : *Admissible diagrams containing a cycle associated with the root systems, Φ_g, but with no proper subsystems of them.*

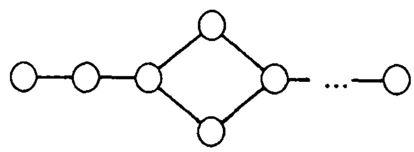
$D_n(a_1)$:



$D_n(a_2)$:

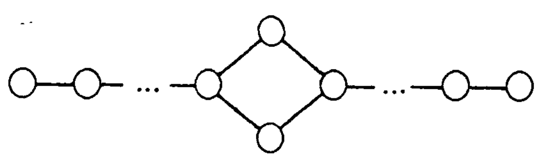


$D_n(a_3)$:



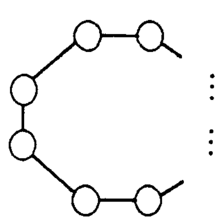
....

$D_n(a_{\frac{n}{2}})$:

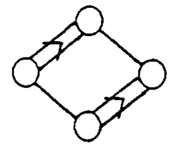


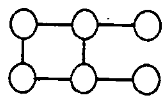
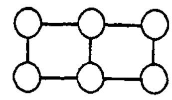
(if n even)

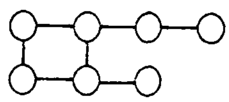
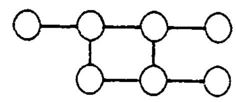
$D_n(b_{\frac{n}{2}})$:

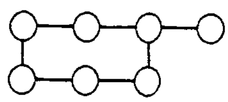
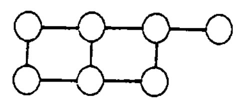


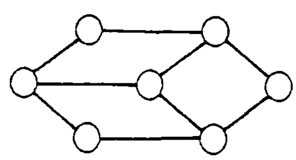
(if n even)

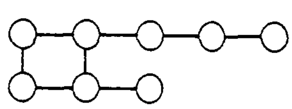
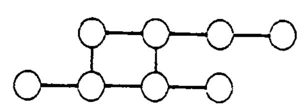
$F_4(a_1)$: 

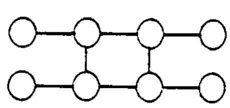
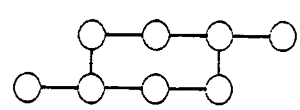
$E_6(a_1)$:  $E_6(a_2)$: 

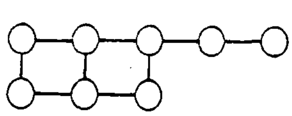
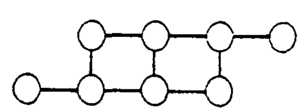
$E_7(a_1)$:  $E_7(a_2)$: 

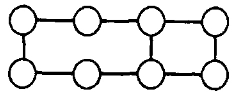
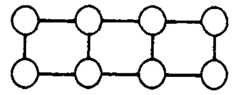
$E_7(b_2)$:  $E_7(a_3)$: 

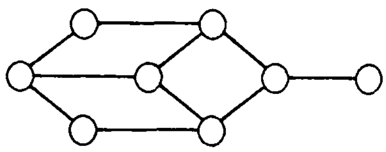
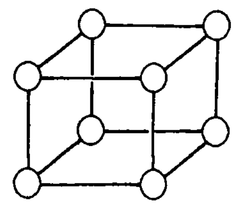
$E_7(a_4)$: 

$E_8(a_1)$:  $E_8(a_2)$: 

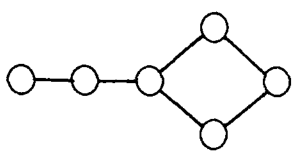
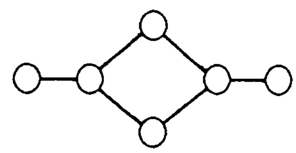
$E_8(a_3)$:  $E_8(b_3)$: 

$E_8(a_4)$:  $E_8(a_5)$: 

$E_8(b_5)$:  $E_8(a_6)$: 

$E_8(a_7)$:  $E_8(a_8)$: 

E.g. The Weyl group W_{D_6} has two admissible diagrams with cycles associated with it but no proper subgroup, namely;

$D_6(a_1)$:  and $D_6(a_2)$: 

Thus from the admissible diagram without cycles $A_2 \times D_6$ we can form the following two diagrams with cycles $A_2 \times D_6(a_1)$ and $A_2 \times D_6(a_2)$.

All graphs corresponding to conjugacy classes are admissible diagrams.

The characteristic polynomial of w on V, the vector space spanned by the simple roots, is determined by its admissible diagram. Each connected admissible diagram has an associated characteristic polynomial when the element w is considered to act solely on the subspace of V spanned by the roots in the admissible diagram. These are listed in Table 4.3. To determine the characteristic polynomial of an arbitrary admissible diagram just multiply the characteristic polynomials of its connected components together with a factor $(t-1)^{\dim V - \dim U}$, where U is the subspace spanned by all the roots in the admissible diagram. This extra term corresponds to the subspace of directions left fixed by w. Note that

$$\dim \mathrm{V} - \dim \mathrm{U} = \operatorname{rank} \mathrm{g} - \text{No. of nodes in admissible diagram.} \tag{4.6}$$

The order of w is equal to the least common multiple (LCM) of the orders of its component parts. A useful result is that;

$$\operatorname{Trace} \mathrm{w} = \operatorname{rank} \mathrm{g} + \text{No. of bonds} - 2(\text{No. of nodes}). \tag{4.7}$$

Every conjugacy class of $\mathrm{W_g}$ determines an admissible diagram but the correspondence is not one to one. A given conjugacy class can be represented by more than one diagram and more than one conjugacy class can have the same diagram.

Table 4.3 : *Characteristic polynomials and eigenvalues of the connected admissible diagrams.*

Admissible diagram	Characteristic polynomial	Order (n)	Eigenvalues
A_n	$t^n+t^{n-1}+.....+t+1$	n+1	(1,2,.....,n)
B_n,C_n	t^n+1	2n	(1,3,5,.....,2n-1)
D_n	$(t^{n-1}+1)(t+1)$	2(n-1)	(n-1,1,3,5,.....,2n-3)
$D_n(a_m)$	$(t^{n-m-1}+1)(t^{m+1}+1)$	2 [m+1,n-m-1]	(x,3x,.....,(2(m+1)-1)x, y,3y,.....,(2(n-m-1)-1)y) x=p/2(m+1) y=p/2(n-m-1)
E_6	$(t^4-t^2+1)(t^2+t+1)$	12	(1,4,5,7,8,11)
$E_6(a_1)$	t^6+t^3+1	9	(1,2,4,5,7,8)
$E_6(a_2)$	$(t^2-t+1)^2(t^2+t+1)$	6	(1,1,2,4,5,5)
E_7	$(t^6-t^3+1)(t+1)$	18	(1,5,7,9,11,13,17)
$E_7(a_1)$	$(t^6-t^5+t^4-t^3+t^2-t+1)(t+1)$	14	(1,3,5,7,9,11,13)
$E_7(a_2),E_7(b_2)$	$(t^4-t^2+1)(t^2-t+1)(t+1)$	12	(1,2,5,6,7,10,11)
$E_7(a_3)$	$(t^4-t^3+t^2-t+1)(t^2-t+1)(t+1)$	30	(1,7,11,13,17,19,23,29)
$E_7(a_4)$	$(t^2-t+1)^3(t+1)$	6	(1,1,1,3,5,5,5)
E_8	$t^8+t^7-t^5-t^4-t^3+t+1$	30	(1,7,11,13,17,19,23,29)
$E_8(a_1)$	t^8-t^4+1	24	(1,5,7,11,13,17,19,23)
$E_8(a_2)$	$t^8-t^6+t^4-t^2+1$	20	(1,3,7,9,11,13,17,19)
$E_8(a_3),E_8(b_3)$	$(t^4-t^2+1)^2$	12	(1,1,5,5,7,7,11,11)
$E_8(a_4)$	$(t^6-t^3+1)(t^2-t+1)$	18	(1,3,5,7,11,13,15,17)
$E_8(a_5),E_8(b_5)$	$t^8-t^7+t^5-t^4+t^3-t+1$	15	(1,2,4,7,8,11,13,14)
$E_8(a_6)$	$(t^4-t^3+t^2-t+1)^2$	10	(1,1,3,3,7,7,9,9)
$E_8(a_7)$	$(t^4-t^2+1)(t^2-t+1)$	12	(1,2,2,5,7,10,10,11)
$E_8(a_8)$	$(t^2-t+1)^4$	6	(1,1,1,1,5,5,5,5)
F_4	t^4-t^2+1	12	(1,5,7,11)
$F_4(a_1)$	$(t^2-t+1)^2$	6	(1,1,5,5)
G_2	t^2-t+1	6	(1,5)

Notation: [a,b] denotes the LCM of a and b.

If ω is a primitive n^{th} root of unity then $(n_1,.....,n_{\text{rank}\,g})$ denotes the set of eigenvalues ω^{n_i}.

4.1.1 Classical Weyl groups:

A_n :

The conjugacy classes of W_{A_n} are in one–to–one correspondence with admissible diagrams of the form:

$$A_{n_1} + A_{n_2} + \ldots\ldots + A_{n_m} \quad \text{such that} \quad \sum_{i=1}^{m}(n_i + 1) = n + 1. \tag{4.8}$$

The order of the corresponding element is: $LCM(n_1+1,\ldots\ldots,n_m+1)$.

Note : A_0 is taken to mean the empty set $\emptyset$.

$W_{A_n} \cong S_{n+1}$ so the conjugacy classes are in one-to-one correspondence with different cycle structures or the partitions of n+1. There are thus $P(n+1)$ conjugacy classes, where $P(n)$ is the **classical partition function** describing the number of different ways a positive integer can be expressed as the sum of positive integers. Explicitly taking the roots of A_n in terms of a set of orthonormal basis vectors, as in the Appendix, the effect of an arbitrary element of W_{A_n} is to permute the basis vectors e_i, i=1,...,n+1.

If an element can be written as the product of disjoint cycles of lengths a,b,c,... we will write its conjugacy class as [a b c ...].

B_n, C_n :

The conjugacy classes of W_{B_n} and W_{C_n} are in one–to–one correspondence with admissible diagrams of the form:

$$A_{n_1}+A_{n_2}+\ldots\ldots+A_{n_r}+C_{m_1}+C_{m_2}+\ldots\ldots+C_{m_s} \quad \text{such that} \quad \sum_{i=1}^{r}(n_i+1)+\sum_{j=1}^{s}m_j = n. \tag{4.9}$$

The order of the corresponding element is: $LCM(n_1+1,\ldots\ldots,n_r+1,2m_1,\ldots\ldots,2m_s)$.

$W_{B_n} \cong W_{C_n} \cong (\mathbb{Z}_2)^n \times S_n$. The effect of an element of $W_{B_n}(W_{C_n})$ is to permute the basis vectors, e_i i=1,...,n, and change the sign of an independent subset of them. Let $s=(k_1,k_2,.....,k_N)$ be an N cycle and $\sigma : \{1,.....,N\} \mapsto \{\pm 1\}$ a particular choice of signs such that

$$e_{k_1} \mapsto \sigma(1)e_{k_2} \mapsto \sigma(1)\sigma(2)e_{k_3} \mapsto ..... \mapsto \sigma(1)...\sigma(N-1)e_{k_N}.$$

If $s^N(e_{k_1}) = e_{k_1}$, that is $\sigma(1)...\sigma(N)=1$, then s is said to be a **positive N cycle** denoted by [N] as before. If $s^N(e_{k_1}) = -e_{k_1}$, that is $\sigma(1)...\sigma(N)=-1$, then s is said to be a **negative N cycle** written as $[\overline{N}]$. [N] is of order N whilst $[\overline{N}]$ is of order 2N. An arbitrary element of $W_{B_n}(W_{C_n})$ can be decomposed into a set of disjoint positive and negative cycles, called its **signed cycle type**. Two elements of the Weyl group are conjugate if and only if they have the same cycle type. Hence the conjugacy classes of $W_{B_n}(W_{C_n})$ are in one–to–one correspondence with the pairs of partitions of μ and λ where $\mu+\lambda = n$; $\mu,\lambda \geq 0$, and the number of conjugacy classes is given by,

$$Q(n) = \sum_{\substack{\mu+\lambda=n \\ \mu,\lambda\geq 0}} P(\mu)P(\lambda).$$

The A_{n_i} admissible diagrams correspond to positive cycles and the C_{n_j} diagrams correspond to negative cycles.

$\underline{D_n}$:

The conjugacy classes of W_{D_n} are in one–to–one correspondence with the signed cycle types with an even number of negative terms except if all the cycles are even and positive in which case there are two conjugacy classes corresponding to the same cycle type. Take $D_n(a_0) = D_n$. The positive i cycle [i] is represented by the admissible diagram A_{i-1} whilst the pair of negative cycles $[\overline{i}\,\overline{j}]$ $i \geq j$ is represented by $D_{i+j}(a_{j-1})$.

The order of an element in the conjugacy class $[n_1,.....,n_r,\overline{m}_1,.....,\overline{m}_s]$ $(s \in 2\mathbb{Z})$ is $LCM(n_1,.....,n_r,2m_1,.....,2m_s)$.

W_{D_n} is a subgroup of W_{C_n} of index 2. An element of W_{C_n} is also an element of W_{D_n} if and only if it changes the sign of an even number of basis vectors e_i. In particular this is equivalent to an element having an even number of negative terms in its signed cycle type.

The number of conjugacy classes is given by

$$\begin{array}{lrl} & \tilde{Q}(n) + P\left(\frac{n}{2}\right) & \text{if } n \in 2\mathbb{Z} \\ \text{or} & \tilde{Q}(n) & \text{if } n \in 2\mathbb{Z}+1, \end{array}$$

where

$$\tilde{Q}(n) = \sum_{\substack{\mu+\lambda=n \\ \mu,\lambda \geq 0}} P(\mu)\tilde{P}(\lambda),$$

and $\tilde{P}(n)$ is the number of partitions of n into an even number of integers.

Table 4.4 : *The first 10 values of* $P(n)$, $\tilde{P}(n)$, $Q(n)$, *and* $\tilde{Q}(n)$.

n	$P(n)$	$\tilde{P}(n)$	$Q(n)$	$\tilde{Q}(n)$
0	1	1	1	1
1	1	0	2	1
2	2	1	5	3
3	3	1	10	5
4	5	3	20	11
5	7	3	36	18
6	11	6	65	34
7	15	7	110	55
8	22	12	185	95
9	30	14	300	150
10	42	22	481	244

4.1.2 Exceptional Weyl groups:

Take all the admissible diagrams associated with W_g and calculate their corresponding characteristic polynomials. For each characteristic polynomial choose one admissible diagram, except for 10 anomalous cases (E_8:1, F_4:8, G_2:1) where two admissible diagrams need to be chosen for a given polynomial. Now all these admissible diagrams correspond to one conjugacy class except for 11 (E_7:6, E_8:5) which correspond to two conjugacy classes. A full list of the conjugacy classes and a choice of admissible diagrams is given in [3]. See also Chapter 7 for those of the simply laced exceptional algebras E_6, E_7 and E_8.

4.2 Conjugacy classes of the automorphism group, aut Φ_g.

The conjugacy classes of the full automorphism groups are not given in [3]. For the cases where there are no diagram symmetries aut $\Phi_g \cong W_g$, so the conjugacy classes are the same. This is true for B_n, C_n, E_7, E_8, F_4 and G_2. Most of the other cases can be determined with the aid of the following theorem.

Theorem (4.2) : If $\Gamma_g \cong \mathbb{Z}_2$ and the inversion automorphism $-\mathbb{1} : \Phi_g \to \Phi_g$ such that $\alpha \mapsto -\alpha$ is an outer automorphism then

$$\mathrm{aut}\,\Phi_g \cong W_g \rtimes \Gamma_g \cong W_g \times \mathbb{Z}_2.$$

In this case if $\{[x]\}$ is the set of conjugacy classes of W_g then $\{[(x,\mathbb{1})],[(x,-\mathbb{1})]\}$ is the set of conjugacy classes of aut Φ_g.

Proof : Let $\mathbb{Z}_2$ denote the group generated by $-\mathbb{1}$ and let $\Theta:\Gamma_g \to \mathbb{Z}_2$ be the isomorphism given by $\Theta(e) = \mathbb{1}$, $\Theta(X) = -\mathbb{1}$. If $-\mathbb{1}$ is outer then there must exist an inner automorphism, $a \in W_g$, such that $aX=-\mathbb{1}$. We must have $a^2 = \mathbb{1}$ because $a^2Xa^{-1} = -\mathbb{1}$, therefore $a^2X^2 = \mathbb{1}$ and we know that $X^2 = \mathbb{1}$. Introduce the map $\psi : \Gamma_g \to W_g$ with $\psi(e)=e$, $\psi(X) = a^{-1} = a$. Now consider the mapping ϕ : aut $\Phi_g \to W_g \times \mathbb{Z}_2$ such that

$$\phi((\sigma, Y)) = (\sigma\psi(Y), \Theta(Y))$$

(i) ϕ is one-to-one because $(\sigma\psi(Y), \Theta(Y)) = (\tau\psi(Z), \Theta(Z))$ implies that $\Theta(Y)=\Theta(Z)$

and hence Z=Y as Θ is one–to–one . Consequently we must have $\sigma = \tau$.

(ii) ϕ is onto as for an arbitrary $\sigma \in \mathrm{W_g}$, $(\sigma, \mathrm{e}) \mapsto (\sigma, \mathbb{1})$ and $(\sigma\mathrm{a}, \mathrm{X}) \mapsto (\sigma, -\mathbb{1})$.

(iii) ϕ is a homomorphism:

$$\phi((\sigma, \mathrm{Y}).(\tau, \mathrm{Z})) = (\sigma \mathrm{Y} \tau \mathrm{Y}^{-1} \psi(\mathrm{YZ}), \Theta(\mathrm{YZ})) \tag{4.10}$$

$$\begin{aligned} \phi((\sigma, \mathrm{Y}))\phi((\tau, \mathrm{Z})) &= (\sigma\psi(\mathrm{Y}), \Theta(\mathrm{Y}))(\tau\psi(\mathrm{Z}), \Theta(\mathrm{Z})) \\ &= (\sigma\psi(\mathrm{Y})\Theta(\mathrm{Y})\tau\psi(\mathrm{Z})\Theta(\mathrm{Y})^{-1}, \Theta(\mathrm{Y})\Theta(\mathrm{Z})) \\ &= (\sigma\psi(\mathrm{Y})\tau\psi(\mathrm{Z}), \Theta(\mathrm{YZ})) \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathrm{Y} = \mathrm{e}: \quad (4.11) &= (\sigma\tau\psi(\mathrm{Z}), \Theta(\mathrm{Z})) = (4.10) \\ \mathrm{Y} = \mathrm{X}: \quad (4.11) &= (\sigma \mathrm{a}^{-1}(\mathrm{aX})\tau(\mathrm{aX})^{-1}\mathrm{a}\psi(\mathrm{YZ}), \Theta(\mathrm{YZ})) \\ &= (\sigma \mathrm{X}\tau \mathrm{X}^{-1}\psi(\mathrm{YZ}), \Theta(\mathrm{YZ})) = (4.10) \end{aligned}$$

Thus ϕ is an isomorphism. Let $\mathrm{Z} = \pm\mathbb{1}$ then $(\sigma, \mathrm{Z})^{-1} = (\sigma^{-1}, \mathrm{Z})$. Hence $(\sigma, \mathrm{Y}) \sim (\tau, \mathrm{Z})$ if and only if $\mathrm{Y} = \mathrm{Z}$ and $\exists \rho \in \mathrm{W_g}$ such that $\rho\sigma\rho^{-1} = \tau$ *i.e.* σ and τ are conjugate in $\mathrm{W_g}$. The conjugacy classes of aut Φ_g are therefore as stated.

Lemma (4.3) : $-\mathbb{1}$ is an inner automorphism if and only if there exist rank g orthogonal vectors in Φ_g. Thus $-\mathbb{1}$ is outer for $\mathrm{A_n}$, $\mathrm{D_n}$ $\mathrm{n} \in 2\mathbb{Z}+1$ and $\mathrm{E_6}$.

Proof : See [3] page 4 (Lemma 4). The following are a choice of rank g orthogonal vectors where they exist. (See Appendix for details of the orthonormal bases):

$\mathrm{B_n}$: $\mathrm{e_i}$ $\mathrm{i} = 1, \ldots\ldots, \mathrm{n}$.

$\mathrm{C_n}$: $2\mathrm{e_i}$ $\mathrm{i} = 1, \ldots\ldots, \mathrm{n}$.

$\mathrm{D_n}$ $(\mathrm{n} \in 2\mathbb{Z})$: $\mathrm{e_1} \pm \mathrm{e_2}$, $\mathrm{e_3} \pm \mathrm{e_4}, \ldots\ldots, \mathrm{e_{n-1}} \pm \mathrm{e_n}$, $\mathrm{i} = 1, \ldots\ldots, \mathrm{n}$.

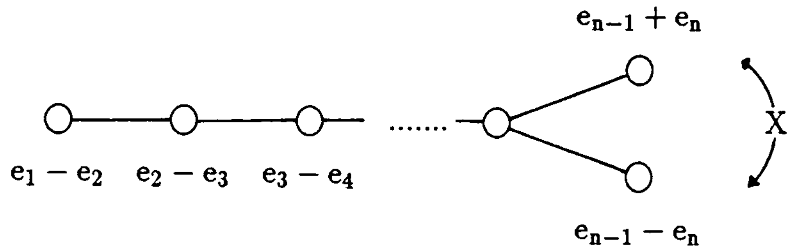
$\mathrm{E_7}$: $\mathrm{e_1} \pm \mathrm{e_2}$, $\mathrm{e_3} \pm \mathrm{e_4}$, $\mathrm{e_5} \pm \mathrm{e_6}$, $\mathrm{e_7} + \mathrm{e_8}$.

$\mathrm{E_8}$: $\mathrm{e_1} \pm \mathrm{e_2}$, $\mathrm{e_3} \pm \mathrm{e_4}$, $\mathrm{e_5} \pm \mathrm{e_6}$, $\mathrm{e_7} \pm \mathrm{e_8}$.

So the only case we have left to worry about is $\operatorname{aut} D_n$ when n is even. Let us consider the case $n > 4$ first. Recall that:

$W_{D_n} = \{$ Signed cycle types with an even no. of negative cycles.$\}$

Consider the Dynkin diagram symmetry, X;



The effect of X is to send $e_n \mapsto -e_n$. Now the transposition $(j\,n) \in W_{D_n}$. So $(j\,n)X(j\,n) \in \operatorname{aut} \Phi_{D_n}$ sends $e_j \mapsto -e_j$ and fixes all the other e_i. So we are able to change the sign of an arbitrary basis vector with an element of $\operatorname{aut} \Phi_{D_n}$. Thus $\operatorname{aut} \Phi_{D_n} \cong W_{B_n}$ $(\cong W_{C_n})$ as it consists of all possible signed cycle types. As a result the conjugacy classes of $\operatorname{aut} \Phi_{D_n}$ are in one–to–one correspondence with the conjugacy classes of W_{B_n}. Notice that this also means that

$$W_{B_n} \cong W_{D_n} \times \mathbb{Z}_2 \text{ for } n \in 2\mathbb{Z}+1. \tag{4.12}$$

Consequently we have the following result:

Lemma (4.4) :

$$Q(n) = \begin{cases} 2\tilde{Q}(n) & \text{for } n \in 2\mathbb{Z}+1, \\ 2\tilde{Q}(n) - P(\frac{n}{2}) & \text{for } n \in 2\mathbb{Z}. \end{cases}$$

Proof :

$n \in 2\mathbb{Z}+1$: Just count the conjugacy classes for the two sides of the automorphism (4.12).

$n \in 2\mathbb{Z}$:

(i) $n = 2, 4$: Explicit calculation.

$\underline{(ii)\ \mathrm{n} > 4:}$ Notice that elements of W_{D_n} which have the same cycle structure but are non-conjugate in W_{D_n} become conjugate in aut Φ_{D_n}. There are $P(\frac{n}{2})$ pairs of such conjugacy classes consisting of those containing only an even number of positive cycles. All the other conjugacy classes in W_{D_n} become two conjugacy classes in aut Φ_{D_n} because for each partition of n there are as many ways to choose an odd number of pieces of this partition to be even cycles as there are to choose an even number. Hence counting the conjugacy classes of the righthand side we have,

$$2\tilde{Q}(\mathrm{n}) - P\left(\frac{\mathrm{n}}{2}\right).$$

It is also true that aut $\Phi_{D_2} \cong W_{B_2} \cong D_4$, where D_4 is the dihedral group of order 8. To see that aut $\Phi_{D_2} \cong D_4$ let X, X′ be the Weyl reflections in the simple roots of the two copies of A_1, and let Y be the outer automorphism that interchanges them

$$\begin{array}{ccccc}
 & \leftarrow \quad X \quad \rightarrow & & \\
-\alpha & \longleftarrow\!\!+\!\!\longrightarrow & \alpha & \Uparrow \\
 & & & Y \\
-\alpha' & \longleftarrow\!\!+\!\!\longrightarrow & \alpha' & \Downarrow \\
 & \leftarrow \quad X' \quad \rightarrow & &
\end{array}$$

Taking A = X, B = YX it is easy to show that $A^2 = B^4 = 1$ and $BA = AB^3$, the defining relations of D_4.

There are $Q(2) = 5$ conjugacy classes: $\{e\}, \{X, X'\}, \{XX'\}, \{Y, YXX'\}, \{YX, YX'\}$.

The full automorphism group for the root system of D_4 is in fact isomorphic to the full automorphism group of the F_4 root system. This can be most clearly understood by realizing that the long and short roots of F_4, separately, are just copies of the D_4 root system, the lattice being scaled up by an appropriate factor of $\sqrt{2}$ for the long roots. As aut Φ_{F_4} must send long roots into long roots and short roots into short roots it must be a subgroup of aut Φ_{D_4}. However the dimensions of the two groups are the same so they must actually be isomorphic

$$\mathrm{aut}\,\Phi_{D_4} \cong \mathrm{aut}\,\Phi_{F_4}. \tag{4.13}$$

The useful thing about his correspondence is that because the Dynkin diagram of F_4 has no symmetries there are no related outer automorphisms and so its automorphism group is simply its Weyl group

$$W_{D_4} \times S_3 \cong W_{F_4}. \tag{4.14}$$

Thus the conjugacy classes of aut Φ_{D_4} are in one–to–one correspondence with those of W_{F_4}. To implement the isomorphism in (4.13) explicitly just note the effect of an automorphism of the F_4 root system on either the long or the short roots to find the corresponding element in aut Φ_{D_4}. We list the conjugacy classes of aut Φ_{D_4} in Table 4.6.

The conjugacy classes of inner automorphisms of aut Φ_{D_4} are just the ones with simply laced admissible diagrams in which case the W_{D_4} admissible diagram is just the W_{F_4} admissible diagram.

Notice that there are two sets of three conjugacy classes in W_{D_4} that become one conjugacy class in aut Φ_{D_4}. These are the ones with admissible diagrams $\{ D_3, A_3', A_3'' \}$ and $\{ D_2, (A_1^2)', (A_1^2)'' \}$ which in turn correspond to cycle types $\{ [\bar{2}1\bar{1}], [4], [4] \}$ and $\{ [11\bar{1}\bar{1}], [22], [22] \}$ respectively. All the other non-conjugate elements in W_{D_4} remain non-conjugate in aut Φ_{D_4}.

It is only for the case of D_n, $n \in 2\mathbb{Z}$, that there exist elements that are not conjugate in W_g but are in aut Φ_g.

Table 4.5 : *Conjugacy classes of* W_{D_4}.

	Cycle types	Admissible diagram	Order	Size of conjugacy class
1	$[1111]$	ϕ	1	1
2	$[11\bar{1}\bar{1}]$	$D_2 \cong A_1^2$	2	6
3	$[\bar{1}\bar{1}\bar{1}\bar{1}]$	$D_2^2 \cong A_1^4$	2	1
4	$[211]$	A_1	2	12
5	$[2\bar{1}\bar{1}]$	$A_1 \times D_2 \cong A_1^3$	2	12
6	$[\bar{2}1\bar{1}]$	$D_3 \cong A_3$	4	24
7	$[22]$	$(A_1^2)'$	2	6
8	$[22]$	$(A_1^2)''$	2	6
9	$[\bar{2}\bar{2}]$	$D_4(a_1)$	4	12
10	$[31]$	A_2	3	32
11	$[\bar{3}\bar{1}]$	D_4	6	32
12	$[4]$	A_3'	4	24
13	$[4]$	A_3''	4	24
				192

Table 4.6 : *Conjugacy classes of* aut Φ_{D_4}.

	Admissible diagram of W_{F_4}	Inner/ Outer	W_{D_4} Admissible diagram	W_{D_4} Cycle types	Order	Size of conjugacy class
1	ϕ	I	ϕ	$[1111]$	1	1
2	A_1	I	A_1	$[211]$	2	12
3	$\tilde{A}_1$	O			2	12
4	A_1^2	I	A_1^2	$[11\bar{1}\bar{1}],[22]$	2	18
5	$A_1 \times \tilde{A}_1$	O			2	72
6	A_2	I	A_2	$[31]$	3	32
7	$\tilde{A}_2$	O			3	32
8	B_2	O			4	36
9	A_1^3	I	A_1^3	$[2\bar{1}\bar{1}]$	2	12
10	$A_1^2 \times \tilde{A}_1$	O			2	12
11	A_3	I	A_3	$[\bar{2}1\bar{1}],[4]$	4	72
12	$B_2 \times A_1$	O			4	72
13	C_3	O			6	96
14	B_3	O			6	96
15	$A_1 \times \tilde{A}_2$	O			6	96
16	$A_2 \times \tilde{A}_1$	O			6	96
17	A_1^4	I	A_1^4	$[\bar{1}\bar{1}\bar{1}\bar{1}]$	2	1
18	$A_2 \times \tilde{A}_2$	O			3	16
19	$A_3 \times \tilde{A}_1$	O			4	36
20	$C_3 \times A_1$	O			6	32
21	D_4	I	D_4	$[\bar{3}\bar{1}]$	6	32
22	$D_4(a_1)$	I	$D_4(a_1)$	$[\bar{2}\bar{2}]$	4	12
23	B_4	O			8	144
24	F_4	O			12	96
25	$F_4(a_1)$	O			6	16
						1152

Note : ~ denotes long roots of F_4 and we identify the short roots with the root system of D_4.

Table 4.7 : *Summary of the autΦ_g which are not isomorphic to their Weyl group W_g.*

g	aut Φ_g	No. of conjugacy classes.
A_n	$W_{A_n} \times \mathbb{Z}_2$	$2P(n+1)$
D_4	W_{F_4}	25
D_n $n \in 2\mathbb{Z}+1$	$W_{D_n} \times \mathbb{Z}_2 \cong W_{B_n}$	$Q(n)$
D_n $n \in 2\mathbb{Z}$, $n \neq 4$	W_{B_n}	$Q(n)$
E_6	$W_{E_6} \times \mathbb{Z}_2$	50

4.3 Matrix representations of aut Φ_g for the classical Lie algebras.

Let $\{e_i\}$ be orthonormal vectors.

$\underline{A_n}$: Φ_{A_n} lies in the n-dimensional subspace of the space spanned by the vectors e_i $i=1,\ldots,n+1$ which is orthogonal to the vector $N=e_1+\ldots+e_{n+1}$. Let w_N be the Weyl reflection in the hyperplane orthogonal to N. This leaves the subspace fixed but sends N into -N. With respect to the orthonormal basis this looks like

$$w_N = \frac{1}{n+1}\begin{pmatrix} n & -1 & -1 & .. & -1 \\ -1 & n & -1 & .. & -1 \\ .. & .. & .. & .. & .. \\ -1 & -1 & -1 & .. & n \end{pmatrix}.$$

w_N necessarily commutes with all the elements of aut Φ_{A_n}. Now consider the group $G \equiv \text{aut}\,\Phi_{A_n} \times \mathbb{Z}_2$ obtained by appending w_N to aut Φ_{A_n}. Let $R: G \rightarrow GL(\mathbb{R}^{n+1})$ be a representation of G. We wish to choose a representation of aut Φ_{A_n} from this R. To do this we need a homomorphism $\phi: G \rightarrow \mathbb{Z}_2$ as then we can choose $\{ R(g, \phi(g)) \mid g \in \text{aut}\,\Phi_{A_n} \}$ as our representation of aut Φ_{A_n} because

$$R(g, \phi(g))R(h, \phi(h)) = R(gh, \phi(gh)) \;\; \forall g, h \in \text{aut}\,\Phi_g.$$

One choice is obtained by taking ϕ to be the identity automorphism, that is

$\phi(g)=e\ \forall g \in G$. However a nicer choice is produced by using the fact that $\mathrm{aut}\,\Phi_{A_n} \cong W_{A_n} \times \mathbb{Z}_2$, as shown in Theorem (4.2) and Lemma (4.3), to construct the homomorphism

$$\phi : G \to \mathbb{Z}_2 \text{ such that } \phi(g) = \begin{cases} e & \text{if g is inner,} \\ w_N & \text{if g is outer.} \end{cases}$$

If g is inner, *i.e.* $g \in W_{A_n}$, we must take R(g,e). Inner automorphisms are in one–to–one correspondence with permutations of the basis vectors e_i. Hence the set $\{R(g,e) \mid g \in W_{A_n}\}$ consists of all $(n+1)\times(n+1)$ matrices with one and only one 1 in each row and column, all the other entries being zero.

The diagram automorphism X is outer so we must pick $R(X,w_N)$ to represent X. It is a simple calculation to see that it has the form

$$\begin{pmatrix} 0 & & -1 \\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & 0 \end{pmatrix}.$$

Now $-R(X,w_N) = R(\tilde{g},e)$ for some $\tilde{g} \in \mathrm{aut}\,\Phi_{A_n}$ as it is a permutation matrix. So $R(\tilde{g}^{-1}X, w_N) = -\mathbb{1}$. Hence the matrices representing outer automorphisms are just $-\mathbb{1}$ times the permutation matrices

$$R(g, w_N) = -R(g\tilde{g}^{-1}X, e) \quad g \in \mathrm{aut}\,\Phi_{A_n}$$

as expected.

$\underline{B_n, C_n:}$ With respect to the orthonormal basis the matrices representing $W_{B_n} \cong \mathrm{aut}\,\Phi_{B_n}$ $(\cong W_{C_n})$ are just the n-dimensional matrices with one and only one ± 1 in each row and column all other entries being zero.

$\underline{D_n:}$

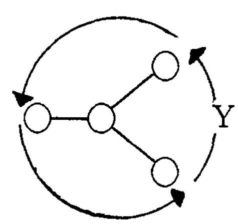
$\underline{(1)\ n \neq 4:}$ Recalling that $\mathrm{aut}\,\Phi_{D_n} \cong W_{B_n}$ and the fact that $\mathrm{aut}\,\Phi_{D_n}$ acts in the same way on the basis vectors we can see that its matrix representation is the same as that for W_{B_n}. The representation of W_{D_n} consists of the subset of matrices containing an even number of minus ones. (Notice that $\phi : R(W_{B_n}) \rightarrow \mathbb{Z}_2$ such that

$$\phi(R(g)) = \begin{cases} 1 & \text{if } R(g) \text{ contains an even no. of -1s,} \\ -1 & \text{if } R(g) \text{ contains an odd no. of -1s.} \end{cases}$$

is a homomorphism).

<u>(2) D_4 :</u> The matrix representation of W_{D_4} is obtained in the same way as that for the other W_{D_n}. However in the case of aut Φ_{D_4} we have in addition to the usual diagram symmetry a rotational symmetry. In terms of the orthonormal basis this is given by

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \; Y^2 = Y^T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$



The full matrix representation consists of the set of 4×4 matrices with one ±1 in each row and column all other entries being zero, multiplied by either $\mathbb{1}_4$, Y or Y^2.

The fact that for $n \in 2\mathbb{Z}$ W_{D_n} contains non-conjugate elements with the same cycle structure can be seen relatively easily with respect to this representation. Let

$$X_n = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 1 & 0 & & & 0 \end{pmatrix} \text{ be an } n \times n \text{ matrix where } n \in 2\mathbb{Z}.$$

It is easy to see that $X_n \in [n]$. Let $I'_n = \text{diag}(1,...,1,-1,..., -1)$ where there are $\frac{n}{2}$ plus and minus ones. Obviously $X'_n \equiv I'_n X_n$ is also of order n and $X'_n \in [n]$. Let

$$A = \begin{pmatrix} 0 & a_1 & & \\ 0 & 0 & a_2 & \\ & & & \ddots \\ a_n & 0 & & 0 \end{pmatrix}$$

$$AX_nA^{-1} = \mathrm{diag}(a_1a_2, a_2a_3, ..., a_na_1)X_n$$

A is the most general matrix which when conjugated with X_n leaves the off-diagonal form of X_n unaltered. So $X_n \sim X_n'$ implies that we have a matrix A with $a_1 = a_2 = ... = a_{\frac{n}{2}+1} = \pm 1$, $a_{\frac{n}{2}+2} = \mp 1, a_{\frac{n}{2}+3} = \pm 1, ..., a_n = \mp 1$. In either case we have an odd number of minus ones so that $A \in [\overline{n}]$ and thus is not an element of W_{D_n}. Hence X_n and X_n' have the same cycle structure but are not conjugate in W_{D_n}.

5. Automorphisms which leave only the origin fixed.

An interesting subset (but not subgroup) of root system automorphisms are the set, $\{\sigma\}$, for which σ and all its powers (not equal to a multiple of its order) leave nothing but the origin fixed. We shall call such automorphisms no fixed point automorphisms or NFPAs. They are a useful subset as they allow simplification of some later calculations (see Chapters 6 and 7). In particular given any automorphism $\sigma \in \text{aut}\,\Phi_{\text{g}}$ we can form the abelian group generated by σ, $\Omega = <\sigma>$, which is isomorphic to $\mathbb{Z}_n$, where n is the order of σ. The number of orbits produced by the action of Ω on Φ_{g} will be $\geq \frac{|\Phi_{\text{g}}|}{n}$ with the equality being obtained if and only if σ is a NFPA. This in turn means that NFPAs produce the smallest possible invariant subalgebra g_0 of g for a given order of twist when $\sigma \in \text{aut}\,\Phi_{\text{g}}$ is extended to an automorphism, $\Sigma \in \text{aut}_{\text{H}}\text{g}$, of the whole algebra (see Chapter 3). Another consequence is that the order of a NFPA must divide $|\Phi_{\text{g}}|$.

In this section we determine all the NFPAs for the root systems of simple Lie algebras.

5.1 Some general results on crystallographic elements that only fix the origin.

Let Λ be a d-dimensional lattice and V the real vector space spanned by elements of Λ. Let $G \subset O(d)$ be the **crystallographic** or **point group** which leaves Λ invariant. With respect to a basis of Λ, which is also a basis for V, an element $\sigma \in G$ must be an integer matrix as it takes the lattice into itself. In particular it must have an integer trace so all its complex eigenvalues, except ± 1, must come in conjugate pairs ω and ω^{-1}. The determinant of σ will be ± 1 depending on whether there are an even or odd number of -1 eigenvalues. In the former case σ is a rotation whilst in the latter it is a reflection.

In addition the fact that an element is crystallographic puts a great restriction on its possible order in a given dimension (see for eg p32 of [42]).

We shall denote the highest common factor of two positive integers a and b by (a,b). Let $\phi(n)$ denote the **Euler function**, the number of integers $0 \leq m \leq n$ which are relatively prime to n, that is $(n, m) = 1$ Also we will use the notation $\text{LCM}(\text{n}_1, ..., \text{n}_r)$ to denote the lowest common multiple of a sequence of positive integers $(\text{n}_1, ..., \text{n}_r)$.

Table 5.1 : *The first 30 values of* ϕ(m).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ϕ(n)	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
ϕ(n)	8	16	6	18	8	11	10	22	8	20	12	18	12	28	8

Note : If p is prime then ϕ(p)=p-1.

A complex number whose n^{th} power is 1 but whose m^{th} power, for $m < n$, is not 1 is called a **primitive n^{th}root of unity.**

Theorem(5.1) (**Crystallographic restriction**) : If σ is an element of a crystallographic subgroup of O(d) of order n then $\phi(n) \leq$d.

Proof : The characteristic equation, det(1-σ)=0, has d roots. This equation has integer coefficients as can be seen by choosing the basis of V which makes σ an integer matrix. Hence the equation will be invariant under permutations of the primitive n^{th} roots of unity. Since at least one primitive n^{th} root of unity must satisfy this equation so must all the $\phi(n)$ such roots. Hence we are led to the restriction d$\geq$ $\phi(n)$.

Table 5.2 : *Table of possible orders of crystallographic elements in* $d \leq$ 9 *dimensions*

Dimension	Possible orders.
2, 3	1 2 3 4 6
4, 5	1 2 3 4 5 6 8 10 12
6, 7	1 2 3 4 5 6 7 8 9 10 12 14 18
8, 9	1 2 3 4 5 6 7 8 9 10 12 14 15 16 18 20 24 30

Note :

1. If n >2 then $\phi(n)$ is even therefore the restriction is the same for d$=$2k and d$=$2k+1, where k is a positive integer.
2. The restriction is satisfied by odd n $\leq$d+1 and even n $\leq$2d. Also odd primes n $>$d+1 will not satisfy the restriction.

For a given n there are $\phi(n)$ primitive roots: ω^m where $(m,n) = 1$ and $\omega = e^{\frac{2\pi i}{n}}$. If all the powers, $m < n$, of σ are to leave nothing but the origin fixed then obviously its eigenvalues must consist solely of primitive roots of unity. In particular -1 is not an allowed eigenvalue for $n > 2$. Hence because all the other eigenvalues, except +1, come in conjugate pairs and the number of eigenvalues is equivalent to the dimension of the vector space: NFPAs of order greater than 2 are only possible in even dimensional spaces (*i.e.* for algebras of even rank).

By the proof of the crystallographic restriction theorem if one of the allowed eigenvalues is present then they all are. Therefore we can only have NFPAs of order n in dimensions $\mathrm{d} \in \phi(n)\mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the positive integers. For example NFP fifth order automorphisms can only occur in dimensions which are a multiple of 4. For each particular dimension such an automorphism is unique up to conjugacy in $\mathrm{O}(\mathrm{d})$. In the complex basis in which it is diagonalised it just consists of the primitive n^{th} roots of unity repeated $\mathrm{d}/\phi(n)$ times down the diagonal.

Let $\mu(n)$ be equal to the sum of the primitive n^{th} roots of unity:

$$\mu(n) \equiv \sum_{\substack{m=1 \\ (m,n)=1}}^{n-1} \omega^m.$$

It can be shown that $\mu(n)$ is in fact equal to the **Möbius function** which is alternatively defined as follows. Write the positive integer $n \in \mathbb{Z}^+$ as a unique product of primes $n = \mathrm{p}_1^{\mathrm{n}_1} \ldots\ldots \mathrm{p}_\mathrm{k}^{\mathrm{n}_\mathrm{k}}$ then

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if any of the } \mathrm{n}_i > 1, \\ (-1)^{\mathrm{k}}, & \text{if all the } \mathrm{n}_i = 1. \end{cases}$$

Let σ be a NFPA of order n acting in a d–dimensional space then its trace is given by:

$$\mathrm{Trace}\,\sigma = \mathrm{d}\frac{\mu(n)}{\phi(n)}.$$

Table 5.3 summarises all the relevant information for NFPAs in nine or less dimensions.

Table 5.3 : *Possible dimensions and traces for all NFPAs that can occur in $d \leq 9$ dimensions.*

n	$\phi(n)$	$\mu(n)$	Allowed dimensions d	Allowed traces
2	1	-1	$\mathbb{Z}^+$	-d
3	2	-1	$2\mathbb{Z}^+$	-d/2
4	2	0	$2\mathbb{Z}^+$	0
5	4	-1	$4\mathbb{Z}^+$	-d/4
6	2	1	$2\mathbb{Z}^+$	d/2
7	6	-1	$6\mathbb{Z}^+$	-d/6
8	4	0	$4\mathbb{Z}^+$	0
9	6	0	$6\mathbb{Z}^+$	0
10	4	1	$4\mathbb{Z}^+$	d/4
12	4	0	$4\mathbb{Z}^+$	0
14	6	1	$6\mathbb{Z}^+$	d/6
15	8	1	$8\mathbb{Z}^+$	d/8
16	8	0	$8\mathbb{Z}^+$	0
18	6	0	$6\mathbb{Z}^+$	0
20	8	0	$8\mathbb{Z}^+$	0
24	8	0	$8\mathbb{Z}^+$	0
30	8	-1	$8\mathbb{Z}^+$	-d/8
n	ϕ(n)	μ(n)	$\phi(n)\mathbb{Z}^+$	$d\frac{\mu(n)}{\phi(n)}$
p = prime	p-1	-1	$(p-1)\mathbb{Z}^+$	$\frac{-d}{(p-1)}$

5.2 INNER AUTOMORPHISMS.

Theorem (5.2) : Let $w \in$ aut Φ_g. If w is a NFPA then:

(*i*) its admissible diagram must have rank g number of spots.

(*ii*) if g is a classical Lie algebra and w has the cycle decomposition $[n_1, ..., n_r, \overline{m}_1, ..., \overline{m}_s]$ and thus of order $n = \text{LCM}(n_1,....,n_r,2m_1,....,2m_s)$ then $n_1 = ... = n_r = 2m_1 = ... = 2m_s = n$.

Proof :

(*i*) : An element corresponding to $k \leq d$ reflections in linearly independent roots in an d-dimensional vector space leaves a (d-k)-dimensional subspace fixed (Lemma (4.1)). So for an element to leave only the origin fixed it must consist of reflections in d = rank g linearly independent roots. The relevant admissible diagrams are those with rank g nodes. Note : This allows us to find all the inner automorphisms of a given root system which leave only the origin fixed. They are those whose admissible diagrams contain rank g nodes. In particular these automorphisms are the product of an even number of reflections and hence **rotations** in an even dimensional space. Likewise they are the product of an odd number of reflections and hence **reflections** in an odd dimensional space. In this latter case all such automorphisms will consequently be of even order.

(*ii*) : Firstly notice that all the classical root systems contain the set of vectors e_i-e_j, $1 \leq i, j \leq$ rank g or (rank g +1). Now let $n_{j+r} = 2m_j$, j=1,...,s. As $n = \text{LCM}(n_1,.....,n_{r+s})$ all the n_i must divide n, that is for each n_i there must exist $\lambda_i \in \mathbb{Z}^+$ such that $n = \lambda_i n_i$. If there is only one cycle then $n = n_1$ trivially so we assume that there is more than one, *i.e.* $r+s > 1$.

Assume that $\lambda_i \neq 1$ and let $(a_1,a_2,...,a_{n_i})$ be the associated cycle. Obviously $w^{\frac{N}{\lambda_i}}$ will fix all the vectors $\{e_{a_k} \mid k = 1, ..., n_i\}$.

(a) $n_i \neq 1$: $w^{\frac{n}{\lambda_i}}$ fixes the vectors e_{a_1}-e_{a_2}.

(b) n_i=1: let e_i be the fixed vector and $(b_1,b_2,...,b_{m_i})$ be any other cycle in the decomposition of w. w fixes the vector

$$\sum_{k=1}^{m}(e_{b_k} - e_i) \in \Phi_g$$

Thus we must have λ_i=1 $\forall$i.

Corollary (5.3) : By examination of the possible Carter diagrams we have *inner* automorphisms of the following orders which leave only the origin fixed;

Algebra	Orders of automorphisms	No. of non-conjugate automorphisms
A_n :	n+1	1
B_n, C_n :	$2\mathrm{LCM}(n_1, \ldots\ldots, n_r)$ where $\sum_{i=1}^{r} n_i = n$	$P(n)$
D_n :	$2\mathrm{LCM}(n_1, \ldots\ldots, n_s)$ where $\sum_{i=1}^{s} n_i = n$ and $s \in 2\mathbb{Z}$	$\tilde{P}(n)$
E_6 :	3, 6(2), 9, 12.	5
E_7 :	2, 4, 6(4), 8, 10, 12, 14, 18, 30.	12
E_8 :	2, 3, 4(2), 5, 6(6), 8(2), 9, 10(2), 12(6), 14, 15, 18(2), 20, 24, 30(2).	30
F_4 :	2, 3, 4, 6(3), 8(2), 12.	9
G_2 :	2, 3, 6.	3

A figure in brackets denotes how many non–conjugate (in the Weyl group) automorphisms of a given order there are.

Classical Lie Algebras :

A_n : The admissible diagram $A_{n_1}+\ldots\ldots+A_{n_m}$ corresponds to reflections in $\sum_{i=1}^{m} n_i$ roots. Thus Theorem (5.2) (i) and (4.8) together imply that

$$n+m = n+1 \qquad i.e.\ m = 1.$$

Therefore the only possible admissible diagram for a NFPA is A_n. However looking at the eigenvalues of such an element we see that they are ω^k where $1 \leq k \leq n$ and ω is a primitive n^{th} root of unity. Thus if no power less than n of this automorphism is to leave a fixed direction then all the integers k must be relatively prime to n+1. Equivalently n+1 must be prime.

$\underline{B_n, C_n:}$ The admissible diagram $A_{n_1}+.....+A_{n_r}+C_{m_1}+.....+C_{m_s}$ corresponds to reflections in $\sum_{i=1}^{r} n_i + \sum_{j=1}^{s} m_j$ roots. Thus Theorem (5.2) (i) and (4.9) together imply that

$$n + r = n \qquad i.e.\ r = 0.$$

In addition Theorem 5.2 (ii) implies that $m_1 = m_2 = ... = m_s \equiv m$. The resulting element is of order 2m. Let $M \equiv n/m$. Such an element has the set of eigenvalues ω^k k=1,3,....,2m-1 where $\omega^{2m} = 1$ and each eigenvalue appears M times. If all the odd numbers $1 \leq k \leq 2m-1$ are relatively prime to 2m then either m is an even integer or 1. Thus NFPAs can only occur when $n = 2^N p$ for some $N \in \mathbb{Z}^+$ and $p \in 2\mathbb{Z}^+ - 1$. In this case conjugacy classes corresponding to NFPAs are of the form.

$\underline{m = 2^{N-r},\ M = 2^r p:}$

Cycle structure	Admissible diagram	Order	Trace	
$[\overline{2^{N-r}}\ \overline{2^{N-r}} ..... \overline{2^{N-r}}]$ $2^r p$ entries	$C_{2^{N-r}}^{2^r p}$	2^{N-r+1}	$-n\delta_{r,N}$	$r = 0, ..., N$

$\underline{\text{Note:}}$ $C_1 \cong A_1$

$\underline{D_n:}$ Recall that $W_{D_n} < W_{C_n}$ such that an element of W_{C_n} is also an element of W_{D_n} if it has an even number of negative cycles in its signed cycle type. The only NFP inner automorphism of W_{C_n} that this restriction rules out is the order 2^{N+1} ($i.e.$ $r = 0$) one. Hence if $n = 2^N p$, where $N \in \mathbb{Z}^+$, $p \in 2\mathbb{Z}^+ - 1$ the conjugacy classes corresponding to NFPAs are of the form.

$\underline{m = 2^{N-r},\ M = 2^r p:}$

Cycle structure	Admissible diagram	Order	Trace	
$[\overline{2^{N-r}}\ \overline{2^{N-r}} ..... \overline{2^{N-r}}]$ $2^r p$ entries	$D_{2^{N-r+1}}^{2^{r-1} p}(a_{2^{N-r}-1})$	2^{N-r+1}	$-n\delta_{r,N}$	$r = 1, ..., N$

$\underline{\text{Note:}}$ We have defined $D_2(a_1) = A_1^2$ to allow this compact notation.

Exceptional Lie Algebras :

We can explicitly examine the eigenvalues of all the exceptional Lie algebra automorphisms which only fix the origin (*i.e.* those with rank g nodes in their admissible diagrams) to see which are NFPAs. We are left with the following.

Table 5.4 : *Table of all the NFPAs of the exceptional Lie algebras.*

g	Admissible diagram	Order	Trace	Det	No. of orbits	No. of conjugate automorphisms
E_6	A_2^3	3	-3	1	24	80
	$E_6(a_2)$	9	0	1	8	5760
E_7	A_1^7	2	-7	-1	63	1
E_8	A_1^8	2	-8	1	120	1
	A_2^4	3	-4	1	80	$2^7.5.7$
	A_4^2	5	-2	1	48	$2^{11}.3^4.7$
	$D_4(a_1)^2$	4	0	1	60	$2^4.3^3.5.7$
	$D_8(a_3)$	8	0	1	30	$2^8.3^4.5^2.7$
	E_8	30	-1	1	8	$2^{13}.3^4.5.7$
	$E_8(a_1)$	24	0	1	10	$2^{11}.3^4.5^2.7$
	$E_8(a_2)$	20	0	1	12	$2^{12}.3^5.5.7$
	$E_8(a_3)$	12	0	1	20	$2^9.3^3.5^2.7$
	$E_8(a_5)$	15	1	1	16	$2^{13}.3^4.5.7$
	$E_8(a_6)$	10	2	1	24	$2^{11}.3^4.7$
	$E_8(a_8)$	6	4	1	40	$2^7.5.7$
F_4	A_1^4	2	-4	1	24	1
	$A_2 \times \tilde{A}_2$	3	-2	1	16	16
	$D_4(a_1)$	4	0	1	12	12
	B_4	8	0	1	6	144
	F_4	12	0	1	4	96
	$F_4(a_1)$	6	2	1	8	16
G_2	$A_1 \times \tilde{A}_1$	2	-2	1	6	1
	A_2	3	-1	1	4	2
	G_2	6	1	1	2	2

(See also Chapter 7 for simply laced exceptional algebras).

5.3 Outer automorphisms.

Now we need to check to see if there are any outer NFPAs. Firstly we shall look at the special case of Φ_{D_4}. Recalling that aut $\Phi_{D_4} \cong W_{F_4}$ it is easy to see that Φ_{D_4} has NFPAs of order 2, 3, 4, 6, 8 and 12. The order 3, 6, 8 and 12 automorphisms are outer, which agrees with the fact that we found Φ_{D_4} only had inner NFPAs of order 2 and 4.

As a NFPA of order n generates NFPAs of all orders that are divisors of n we can put these automorphisms into two families: one generated by an element of order 12 and the other by an element of order 8.

$$
\begin{array}{ccccccccccc}
A & & & & A^2 & & A^3 & & A^4 & & A^6 \\
12 & \longmapsto & \longmapsto & \longmapsto & 6 & \longmapsto & 4 & \longmapsto & 3 & \longmapsto & 2 \\
\mathbf{OUT} & & & & \mathbf{OUT} & & \mathbf{IN} & & \mathbf{OUT} & & \mathbf{IN} \\
 & & B & & & & B^2 & & & & B^4 \\
 & & 8 & \longmapsto & \longmapsto & \longmapsto & 4 & \longmapsto & \longmapsto & \longmapsto & 2 \\
 & & \mathbf{OUT} & & & & \mathbf{IN} & & & & \mathbf{IN}
\end{array}
$$

OUT denotes an outer automorphism.

IN denotes an inner automorphism.

<u>Note :</u> $A^3 \sim B^2$, $A^6 \sim B^4 \sim -\mathbb{1}_4$ where $\sim$ denotes conjugacy.

Henceforth we only need to look at automorphism groups which are the semi-direct product of a Weyl group and $\mathbb{Z}_2$. For such groups the product of two outer automorphisms is inner so that the outer automorphisms are always of even order and hence their square will always be inner automorphisms of half their order. Consequently we need only check whether there are NFPAs of the following orders;

<u>(1)</u> 2. The only order two NFPA is the reflection in the origin, $\alpha \mapsto -\alpha \quad \forall \alpha \in \Phi_g$. As all root systems have this symmetry there is only going to be a second order outer NFPA if this inversion is outer. That is Φ_g does not contain rank g orthogonal vectors

<u>(2)</u> Twice the orders of known inner NFPAs.

Let us firstly look at the automorphism groups which are a direct product of the Weyl group with $\mathbb{Z}_2$. In this case the inversion automorphism is always outer (Theorem (4.2)).

If $(\sigma, \mathbb{1})$ is an inner automorphism of order n with eigenvalues ω^{n_i}, where $i=1,\ldots..,\mathrm{rank}\, g$ and ω is a primitive n^{th} root of unity, then $(\sigma, -\mathbb{1})$ will be an automorphism of order $2n$ with eigenvalues $-\omega^{n_i}$. Now $(-\omega^{n_i})^k \neq 1$ for $1 \leq k \leq 2n\text{-}1$ implies that n is odd. For if $n = 2m$ then $(-\omega^{n_i})^n = (-1)^{2m}(\omega^{n_i})^n = 1$ and $1 \leq n \leq 2n\text{-}1$. In addition we must have $(\omega^{n_i})^{2m} \neq 1$ for $1 \leq m \leq p\text{-}1$ which means that $(\sigma, \mathbb{1})$ must be a NFPA. Hence $(\sigma, -\mathbb{1})$ will be a NFPA if and only if $(\sigma, \mathbb{1})$ is an odd order NFPA. Let us denote the outer automorphism given by $-\mathbb{1}$ times the inner automorphism obtained from the Carter diagram CD by (CD, $-\mathbb{1}$).

$\underline{A_n:}$

(1): $-\mathbb{1}_n$ is outer.

(2): If n+1 is prime then there is an outer NFPA of order 2(n+1) *i.e.* $(A_n, -\mathbb{1})$.

$\underline{D_n\ (n \in 2\mathbb{Z}+1):}$

(1): $-\mathbb{1}_n$ is outer.

(2): There are no other outer NFPAs.

$\underline{E_6:}$

(1): $-\mathbb{1}_n$ is outer.

(2): There are outer NFPAs of order 6 and 18 *i.e.* $(A_2^3, -\mathbb{1})$ and $(E_6(a_1), -\mathbb{1})$.

Finally let us look at

$\underline{D_n\ (n \in 2\mathbb{Z},\ n \neq 4):}$

(1): $-\mathbb{1}_n$ is inner.

(2): NFPAs are the same as those for $W_{B_n}(W_{C_n})$. Therefore if $n = 2^N p$ then there is an outer NFPA of order 2^{N+1}.

5.4 Summary.

In summary we have found the following automorphisms of the root systems of simple Lie algebras whose powers either leave no points but the origin fixed or are equal to the identity automorphism. This is a complete list up to conjugation in the respective automorphism groups.

CLASSICAL LIE ALGEBRAS :

A_n :	$[2(\widehat{n+1})]$	$\mapsto$	$n+1$	$\mapsto$	$\hat{2}$						$n+1$ prime
	$[\hat{2}]$										$n+1$ not prime
B_n, C_n :	$[2^{N+1}]$	$\mapsto$	2^N	$\mapsto$	....	$\mapsto$	2				$n = 2^N p$
D_n :	$[\widehat{2^{N+1}}]$	$\mapsto$	2^N	$\mapsto$	....	$\mapsto$	2				$n = 2^N p \; N \neq 4$
	$[\hat{2}]$										$n = p$
D_4 :	$[\widehat{12}]$	$\mapsto$	$\hat{6}$	$\mapsto$	4	$\mapsto$	$\hat{3}$	$\mapsto$	2		
	$[\hat{8}]$	$\mapsto$	$\mapsto$	$\mapsto$	4	$\mapsto$	$\mapsto$	$\mapsto$	2		$p \in 2\mathbb{Z}+1$

$D_3 \cong A_3$

EXCEPTIONAL LIE ALGEBRAS :

E_6 :	$[\widehat{18}]$	$\mapsto$	9	$\mapsto$	$\hat{6}$	$\mapsto$	3	$\mapsto$	$\hat{2}$							
E_7 :	[2]															
E_8 :	[30]	$\mapsto$	15	$\mapsto$	10	$\mapsto$	$\mapsto$	6	$\mapsto$	5	$\mapsto$	$\mapsto$	$\mapsto$	3	$\mapsto$	2
	[24]	$\mapsto$	$\mapsto$	12	$\mapsto$	8	$\mapsto$	6	$\mapsto$	$\mapsto$	$\mapsto$	4	$\mapsto$	3	$\mapsto$	2
	[20]	$\mapsto$	$\mapsto$	$\mapsto$	10	$\mapsto$	$\mapsto$	$\mapsto$	$\mapsto$	5	$\mapsto$	4	$\mapsto$	$\mapsto$	$\mapsto$	2
F_4 :	[12]	$\mapsto$	6	$\mapsto$	4	$\mapsto$	3	$\mapsto$	2							
	[8]	$\mapsto$	$\mapsto$	$\mapsto$	4	$\mapsto$	$\mapsto$	$\mapsto$	2							
G_2 :	[6]	$\mapsto$	3	$\mapsto$	2											

Notes :

1. The numbers appearing in the above lists denote the orders of the automorphisms.
2. [n] denotes an automorphism which can be used to generate the lower order automorphisms in the same sequence. This is done by taking an appropriate power of it.
3. ˆ denotes that the automorphism is outer.
4. For a given algebra automorphisms of the same order which appear in different sequences are conjugate.

Aside : In [16] it is stated that for an inner automorphism $\sigma \in W_g$ we have $\det(1-\sigma) = n \det A$, where $n \in \mathbb{Z}^+ + \{0\}$ and A is the Cartan matrix of g. If σ leaves some points fixed then obviously $n = 0$. Recall that

$$\det A = \left|\frac{\Lambda_W}{\Lambda_R}\right| = |Z_g|,$$

where Z_g is the centre of g. An automorphism is said to be **non–degenerate** if $\det(1-\sigma) \neq 0$ and **primitive** if $\det(1-\sigma) = \det A$ [16]. The number of inner primitive elements for simply laced algebras is equal to the number of orbits of the extended Dynkin diagram of g, namely 1, $[\frac{n}{2}]$, 3, 5 and 9 for A_n, D_n, E_6, E_7, and E_8 respectively. The primitive elements correspond to the Coxeter element, whose Carter diagram is just the Dynkin diagram of g, and the elements created by the exceptional Carter diagrams associated with g. The numbers of non–degenerate elements are given in Corollary (5.3) p 97.

5.5 Third order NFPAs.

Altogether there are six third order NFPAs upto conjugacy, they are given in Table 5.5.

Table 5.5 : *Third order NFPAs.*

g	A_2	D_4	E_6	E_8	F_4	G_2
g_0	$\mathbf{R}^2$	A_2	A_2^3	A_8	A_2^2	$A_1 \otimes \mathbf{R}$
Inner\Outer	I	O	I	I	I	I
c_σ	1	3	3	9	-	-
F	3	9	27	81	9	3

Notice that:

1. The number of orbifold fixed points (see Chapter 7) is given by; $F = \det(1-\sigma) = 3^{\frac{\text{rank}\, g}{2}}$.

2. They only occur for even rank algebras as there are no third order reflections and they are of necessity the product of rank g reflections.

Four of these automorphisms are related to the triality of D_4 (D_4, E_6, E_8, F_4) whilst the other two (A_2, G_2) are related to the 'triality' rotational symmetry of the A_2 root lattice which is in turn related to the fact that the centre of A_2 is $\mathbb{Z}_3$. We shall look at the former automorphisms in a little more detail.

With respect to the orthonormal basis the root system of D_4 is given by $\pm e_i \pm e_j$ $1 \leq i, j \leq 4$, $i \neq j$. The vector, spinor and $\overline{\text{spinor}}$ weights are given by;

<u>vector :</u> $\pm e_i \quad i = 1, ..., 4$.

<u>spinor :</u> $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ where there are an *odd* number of plus signs.

<u>$\overline{\text{spinor}}$:</u> $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ where there are an *even* number of plus signs.

<u>Note :</u> The D_4 root system is isomorphic to the vector, spinor and $\overline{\text{spinor}}$ weights rotated and scaled up.

With respect to this basis outer third order automorphisms take the form of a 4×4 matrix whose rows are orthogonal spinors ($\overline{\text{spinors}}$) and whose columns are orthogonal $\overline{\text{spinors}}$ (spinors). As they are orthogonal and third order they must satisfy the additional constraint that $M^2 = M^{-1} = M^T$. They have the effect of cyclically permuting the vector, spinor and $\overline{\text{spinor}}$ (or vector, $\overline{\text{spinor}}$ and spinor) weights,

$$\hookrightarrow 8_v \rightarrow 8_s \rightarrow 8_{\bar{s}} \circlearrowleft \;(\text{or} \hookrightarrow 8_v \rightarrow 8_{\bar{s}} \rightarrow 8_s \circlearrowleft)\,.$$

This magic only works for D_4 where it is known as **triality**. There are 48 such automorphisms which split into 2 conjugacy classes. The first contains 32 elements which are conjugate to the third order diagram automorphisms (conjugacy class no. 7). The second (conjugacy class no. 18) consists of the subset of outer third order automorphisms with $-\frac{1}{2}$s down the diagonal. These are the 16 third order NFPAs.

<u>E.g.</u>

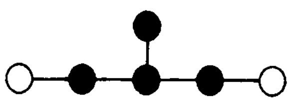
$$M_1 = \begin{pmatrix} -1, & 1, & 1, & 1 \\ -1, & -1, & -1, & 1 \\ -1, & -1, & 1, & -1 \\ -1, & 1, & -1, & -1 \end{pmatrix}$$

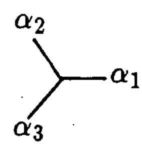
fixes $e_2 - e_4$ and is conjugate to the third order diagram automorphism whilst,

$$M_2 = \begin{pmatrix} -1, & 1, & 1, & 1 \\ -1, & -1, & -1, & 1 \\ -1, & 1, & -1, & -1 \\ -1, & -1, & 1, & -1 \end{pmatrix}$$

is a third order NFPA. Notice that for NFPAs orthogonality implies that $a_{ij} = -a_{ji}$ for $i \neq j$.

We can decompose the roots of E_6 in terms of a D_4 root lattice and an A_2 root lattice scaled down so that the simple roots are of length squared one. Let α_1, α_2 be these scaled down simple roots and let $\alpha_3 = -(\alpha_1 + \alpha_2)$.





The roots of E_6 are then

$$(\mathbf{28}, 0) \oplus (\mathbf{8}_v, \pm\alpha_1) \oplus (\mathbf{8}_s, \pm\alpha_2) \oplus (\mathbf{8}_{\bar{s}}, \pm\alpha_3) \oplus (\mathbf{1}, \alpha_1) \oplus (\mathbf{1}, \alpha_2).$$

The effect of the NFPA is to simultaneously perform a third order NFPA on the D_4 and A_2 components separately. See also [36].

Similarly we can decompose E_8 in terms of a $D_4 \oplus D_4$ subalgebra as

$$(\mathbf{28}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{28}) \oplus (\mathbf{8}_v, \mathbf{8}_v) \oplus (\mathbf{8}_s, \mathbf{8}_s) \oplus (\mathbf{8}_{\bar{s}}, \mathbf{8}_{\bar{s}}).$$

This is just the decomposition of E_8 used in the transcendental fermionic construction of the Kac–Moody algebra [43].

Finally as we stated earlier the root system of F_4 consists of the D_4 root system and the non–zero components of a D_4 root system scaled up so that the simple roots have length squared four.

6. Twisted Vertex Operator Representations of Kac-Moody Algebras.

In this chapter we shall show how it is possible to construct a general twisted vertex operator basic representation of an arbitrarily graded simply–laced Kac–Moody algebra. It is based largely on [5] but similiar results can be found in [16,44]. A general treatment is given in [19]. We start by reviewing the well known [21,22] vertex operator representation of the homogeneous gradation of a simply-laced Kac–Moody algebra, which is now reduced to being a special case. We go on to look at the general construction before explaining the details of the zero–mode space where most of the subtlety lies. We finish by giving some examples of the construction in action.

6.1 Ordinary vertex operator representations.

Let g be a simply laced Lie algebra with root system Φ_g and root lattice Λ_R. Let V be the real span of the roots. Now assume that we have a set of oscillators $\{a_n^i \mid i = 1, ..., \mathrm{rank}\, g\}$ satisfying the **Heisenberg algebra**

$$[a_m^i, a_n^j] = m\delta^{ij}\delta_{m+n,0} \,.$$

That is they form a representation of a graded Cartan subalgebra of a rank g–dimensional Kac-Moody algebra. Let $\mid 0\rangle$ be a vacuum vector for these oscillators and F the Fock space representation of the Heisenberg algebra spanned by the oscillators and the identity operator.

We identify $p = a_0$ as a momentum operator and introduce a corresponding conjugate position operator q with

$$[q^i, p^j] = i\delta^{ij} \,. \tag{6.1}$$

q commutes with all the other oscillators. Let $\mid 0\rangle_q$ be a vacuum vector for q and let $C(\Lambda_R)$ be the infinite–dimensional vector space spanned by the momentum eigenstates, $\mid \alpha\rangle$ such that $\alpha \in \Lambda_R$. $C\ (\Lambda_R)$ is known as the **zero–mode space**. The **translation**

group, Λ_R, is represented on $\mathbf{C}(\Lambda_R)$ by operators of the form, $e^{i\alpha.q}$, $\alpha \in \Lambda_R$.

$$e^{i\alpha.q} \mid \beta\rangle = \mid \alpha+\beta\rangle \qquad \alpha, \beta \in \Lambda_R.$$

Next we form the combined Hilbert space

$$\begin{aligned} H &\equiv F \otimes \mathbf{C}(\Lambda_R), \\ &= \bigoplus_{\alpha \in \Lambda_R} F_\alpha, \end{aligned}$$

where F_α is a Fock space representation of the Heisenberg algebra, isomorphic to F, with a vacuum vector $\mid 0\rangle_q \otimes \mid \alpha\rangle$ which we will write as $\mid 0, \alpha\rangle$.

The **Fubini–Veneziano field** [45] is given by

$$Q^i(z) \equiv q^i - ip^i \ln z + i \sum_{n \neq 0} \frac{a_n^i}{n} z^{-n}.$$

We also introduce the corresponding momentum field

$$P(z) \equiv iz\frac{dQ}{dz} = \sum_{n \in Z} a_n z^{-n}.$$

Finally we are in a position to give a definition of the **vertex operator**, $V(\alpha,z)$ [21,22,46],

$$V(\alpha, z) \equiv : e^{i\alpha.Q(z)} : C_\alpha,$$

The colons : : denote a normal ordering introduced to eliminate the ambiguity in the ordering of non-commuting operators. It is definition is given in Chapter 2, page 32.

The $\{C_\alpha\}$ are a set of operators, variously known in the literature as **Klein factors** or **cocycle operators**, which multiply the momentum eigenstates by an appropriate phase,

$$C_\alpha \mid \beta\rangle = \varepsilon(\alpha, \beta) \mid \beta\rangle \quad \forall\, \beta \in \Lambda_R.$$

where $\varepsilon : \Lambda_R \times \Lambda_R \to \{\pm 1\}$.

We choose ε to be a 2–cocycle,

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma)\,. \tag{6.2}$$

so that the operators $\hat{C}_\alpha \equiv e^{i\alpha.q}C_\alpha$ form a projective representation of Λ_R on $C(\Lambda_R)$ with the factor set ε *i.e.*

$$\hat{C}_\alpha\hat{C}_\beta = \varepsilon(\alpha,\beta)\hat{C}_{\alpha+\beta}\,. \tag{6.3}$$

In general we find that

$$\begin{aligned} V(\alpha,z)V(\beta,\zeta) &= F(\alpha,\beta,z,\zeta) && |\,\zeta\,|<|\,z\,|\,, \\ V(\beta,\zeta)V(\alpha,z) &= S(\alpha,\beta)\frac{\varepsilon(\beta,\alpha)}{\varepsilon(\alpha,\beta)}F(\alpha,\beta,z,\zeta) && |\,z\,|<|\,\zeta\,|\,. \end{aligned} \tag{6.4}$$

where $F(\alpha,\beta,z,\zeta)$ is some appropriate function. $S(\alpha,\beta)$ arises from the interchange of the non–cocycle operator pieces of the vertex operator whilst $\frac{\varepsilon(\beta,\alpha)}{\varepsilon(\alpha,\beta)}$ is due to the reordering of the cocycle operators. In the string picture $S(\alpha,\beta)$ is determined by the boundary conditions of the emitting string.

So we wish, and are free, to choose the factor set to have a specific symmetry under the interchange of its arguments so that the operators $\hat{C}_\alpha$, $\hat{C}_\beta$ in (6.4) pick up the correct phase when their order is swopped, *i.e.*

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = S(\alpha,\beta)\,. \tag{6.5}$$

and

$$\hat{C}_\alpha\hat{C}_\beta = S(\alpha,\beta)\hat{C}_\beta\hat{C}_\alpha\,. \tag{6.6}$$

Now (6.3) implies that

$$S(\alpha+\beta,\gamma) = S(\alpha,\gamma)S(\beta,\gamma) \tag{6.7}$$

which is also consistent with (6.2), whilst (6.6) implies

$$S(\alpha,\beta)S(\beta,\alpha) = 1\,. \tag{6.8}$$

and

$$S(\alpha,\alpha) = 1 \qquad \forall\,\alpha\in\Lambda_R\,. \tag{6.9}$$

Thus $S:\Lambda_R\times\Lambda_R \to \mathbb{C}-\{0\}$ is an alternating bimultiplicative function called a **commutator map** by [5] and a **symmetry factor** by [47].

For the ordinary vertex operator representation, or untwisted bosonic string

$$S(\alpha,\beta) = (-1)^{\alpha.\beta},$$

as in this case we find that

$$F(\alpha,\beta,z,\zeta) = \varepsilon(\alpha,\beta)(z-\zeta)^{\alpha.\beta} : e^{i(\alpha.Q(z)+\beta.Q(\zeta))} : C_{\alpha+\beta}\,.$$

If we take the integral moments of the vertex operators

$$E^n_\alpha \equiv \frac{1}{2\pi i}\oint \frac{dz}{z} z^{n+\frac{\alpha^2}{2}} V(\alpha,z)\,.$$

then

$$\begin{aligned}[E^m_\alpha, E^n_\beta] &= \frac{1}{(2\pi i)^2}\left\{ \oint\limits_{|z|>|\zeta|} \frac{dz}{z} z^m \oint \frac{d\zeta}{\zeta}\zeta^n - \oint\limits_{|z|<|\zeta|} \frac{dz}{z} z^m \oint \frac{d\zeta}{\zeta}\zeta^n \right\} z^{\frac{\alpha^2}{2}}\zeta^{\frac{\beta^2}{2}} F(\alpha,\beta,z,\zeta)\,,\\ &= \frac{1}{(2\pi i)^2}\oint\limits_0 \frac{d\zeta}{\zeta}\zeta^{n+\frac{\beta^2}{2}} \oint\limits_\zeta \frac{dz}{z} z^{m+\frac{\alpha^2}{2}} F(\alpha,\beta,z,\zeta)\,,\end{aligned}$$

where the z contour now encircles ζ positively once whilst the ζ contour encircles the origin positively once [26].

Altogether we have the following commutation relations

$$[E^m_\alpha, E^n_\beta] = \begin{cases} \varepsilon(\alpha,\beta)E^{m+n}_{\alpha+\beta} & \alpha+\beta \in \Phi_g\,, \\ \alpha.a_{m+n} + m\delta_{m+n,0} & \alpha+\beta = 0\,, \\ 0 & \alpha+\beta \notin \Phi_g\,, \end{cases} \tag{6.10}$$

and

$$[a^i_m, a^j_n] = m\delta^{ij}\delta_{m+n,0}\,, \tag{6.11}$$

$$[a^i_m, E^n_\alpha] = \alpha^i E^{m+n}_\alpha\,, \tag{6.12}$$

Now the ε form a 2–cocycle associated with Λ_R which satisfy

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = (-1)^{\alpha.\beta},$$

so by (6.10)–(6.12) the integral moments of the vertex operators defined on the root lattice together with the oscillators and the identity form a basic level one representation of the simply–laced untwisted Kac–Moody algebra $\hat{g}$.

The easiest way to construct the C_α without introducing any extra degrees of freedom is to take them to be functions of momentum. In this case we require [46],

$$\begin{aligned} C_\alpha &= \varepsilon(\alpha, p)\,, \\ &= \sum_{\beta\in\Lambda_R} \varepsilon(\alpha,\beta) \mid \beta+\bar{p}\rangle\langle\beta+\bar{p}\mid , \end{aligned}$$

where $\bar{p}$ is the ground state momentum. We shall look at the construction of such operators in more detail in Section 6.3.

The Virasoro algebra naturally associated with this construction is that obtained by taking integral moments of the energy–momentum tensor or **Virasoro field** [48],

$$L(z) = \frac{1}{2} : P^2(z) := \sum_{n\in\mathbf{Z}} L_n z^{-n}\,,$$

that is

$$L_n = \frac{1}{2}\sum_{m\in\mathbf{Z}} : a^i_{n-m} a^i_m :\,,$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}\,.$$

This is a special case of the Sugarawa construction for vertex operator representations, for more details see [46]. The Virasoro algebra has a natural semi–direct product structure with the Kac–Moody algebra, *i.e.*

$$[L_m, E^n_\alpha] = -nE^{m+n}_\alpha,\ [L_m, a^i_n] = -na^i_{m+n},\ [L_m, 1] = 0.$$

The natural derivation of the Kac–Moody algebra is $d = L_0$. The zero graded subalgebra, that is elements that commute with L_0, form an adjoint representation of the Lie algebra g. The other eigenspaces of L_0 form representations of g. If we define the **partition function** as the generating function for the number of states at each level of the Kac–Moody representation then,

$$P(q) = \frac{\sum_{\alpha\in\Lambda_R} q^{\frac{\alpha^2}{2}}}{\prod_{n=1}^{\infty}(1-q^n)^{\mathrm{rank}\,g}}\,.$$

6.2 TWISTED VERTEX OPERATOR REPRESENTATIONS.

Let $\sigma \in \mathrm{aut}\,\Phi_g$ be a root system automorphism of order n and let $\Sigma = (\sigma, \psi) \in \mathrm{aut}_H^{[\tau]} g$ be the corresponding Lie algebra automorphism of order N. Assume we have the same set of integrally graded oscillators as in the previous section, namely $\{a_n^i \mid i = 1, ..., \mathrm{rank}\, g,\ n \in \mathbb{Z}\}$.

One way to consider the previous untwisted vertex operator construction is as follows. Firstly we take a representation of a $\widehat{U(1)}^{\mathrm{rank}\, g}$ Kac–Moody algebra, given by the moments of the momentum field P(z) and the identity operator, and then extend it to a representation of the homogeneous gradation of the untwisted Kac–Moody algebra $g^{(1)}$. The $\widehat{U(1)}^{\mathrm{rank}\, g}$ algebra thus becomes a graded CSA or **Heisenberg subalgebra** of $g^{(1)}$. The extension is performed by 'integrating' the momentum field P(z) to form the Fubini–Veneziano field Q(z) and then exponentiating this to form the vertex operators. Similarly in the twisted case we take a representation of a different gradation of a $\widehat{U(1)}^{\mathrm{rank}\, g}$ Kac–Moody algebra and extend it in an analogous way to the corresponding **twisted vertex operator representation** of $g^{(\tau)}$ with this different gradation. This is done by constructing the twisted vertex operators to be of the form, $V_\sigma(\alpha, z) = \sum_{r \in Z + \frac{1}{N}} E_\alpha(Nr) z^{-r}$.

$$L\left(U(1)^{\mathrm{rank}\, g}, \mathbb{1}\right) \xrightarrow{\text{vertex operators}} L(g, \mathbb{1}),$$
$$L\left(U(1)^{\mathrm{rank}\, g}, \sigma\right) \xrightarrow{\text{twisted vertex operators}} L(g, \sigma).$$

Firstly we need the new **twisted momentum field**. A suitable choice is given by (where we have suppressed indices)

$$P_\sigma(z) \equiv \frac{1}{N}\left\{P(z^{\frac{1}{N}}) + \sigma^{-1}\left[P(\omega z^{\frac{1}{N}})\right] + ... + \sigma^{-(N-1)}\left[P(\omega^{N-1} z^{\frac{1}{N}})\right]\right\}$$

<u>**Fock space for the Heisenberg subalgebra.**</u>

Let us relabel the oscillators so that

$$b_n^i \equiv \sum_{r=0}^{N-1} \omega^{-nr} (\sigma^{-r})^i{}_j a_n^j \qquad n \in \mathbb{Z}. \tag{6.13}$$

It is sometimes useful to change the indexing on the oscillators to be drawn from $\frac{1}{N}\mathbb{Z}$

in which case we write

$$c^i_r = b^i_{Nr} \qquad r \in \frac{1}{N}\mathbb{Z}.$$

Now the oscillators b^i_n are **creation operators** if $n < 0$ and **annihilation operators** if $n > 0$. If $a^{i\,\dagger}_n = a^i_n$ then $b^{i\,\dagger}_n = b^i_n$. The Fock space, F^σ, for such a representation consists of the complex span of all the states of the form

$$b^{i_1}_{n_1} b^{i_2}_{n_2} \ldots\ldots b^{i_r}_{n_r} \mid 0\rangle \quad n_i < 0$$

where $\mid 0\rangle$ is the same vacuum as for the a^i_n oscillators.

The projection (6.13) removes many linear combinations of the oscillators so that we are left with a subspace of the original, untwisted, Hilbert space F which is left invariant under the representation of the Kac–Moody automorphism $\widehat{\Sigma}$ on F.

To see this a little more clearly let us make a change of basis to one in which σ is diagonal. We assume first that we are already in an orthonormal basis in which the orthogonal transformation σ is in the canonical block form

$$\begin{pmatrix} \mathbb{1}_m & & & & & & \\ & -\mathbb{1}_n & & & & & \\ & & \cos\theta_1 & \sin\theta_1 & & & \\ & & -\sin\theta_1 & \cos\theta_1 & & & \\ & & & & \ddots & & \\ & & & & & \cos\theta_r & \sin\theta_r \\ & & & & & -\sin\theta_r & \cos\theta_r \end{pmatrix}$$

where $\mathbb{1}_m$ is the $m \times m$ identity matrix and $m + n + 2r = \operatorname{rank} g$. We can always choose such a basis. Let the oscillators a^i_n then be given with respect to this basis and let $M = m+n$.

To procced we must consider V as being embedded in the complexification of V *i.e.* $V \otimes \mathbb{C}$. To diagonalise σ we conjugate it by the *complex* matrix A which has the block diagonal form

$$
\mathrm{A} \equiv \begin{pmatrix} \mathbb{1}_{\mathrm{M}} & & & \\ & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \end{pmatrix}
$$

Thus $\mathrm{A}\sigma\mathrm{A}^{-1} = \mathrm{diag}(\omega^{\mathrm{n}_1}, \ldots\ldots, \omega^{\mathrm{rank}\,\mathrm{g}})$ where ω is a primitive N^{th} root of unity and ω^{n_i} are the complex eigenvalues of σ. Notice that the last $2r$ eigenvalues come in conjugate pairs *i.e.* $\mathrm{n}_r = \mathrm{N} - \mathrm{n}_{r+1}$ and $\omega^{\mathrm{n}_r} = \omega^{-\mathrm{n}_{r+1}}$ $\quad r = \mathrm{M},\ \mathrm{M}+2, \ldots\ldots, \mathrm{rank}\,\mathrm{g}$.

Let $\tilde{\alpha}^i = \mathrm{A}^i{}_j\,\alpha^j$ and $\tilde{\mathrm{a}}^i_{\mathrm{n}} = \mathrm{A}^i{}_j\,\mathrm{a}^j_{\mathrm{n}}$. If the i^{th} eigenvalue is ω^{n_i} then with respect to this new basis

$$
\begin{aligned}
\tilde{\mathrm{c}}^i_r \equiv \tilde{\mathrm{b}}^i_{\mathrm{N}r} &= \frac{1}{\mathrm{N}}\sum_{s=0}^{\mathrm{N}-1}(\omega^{\mathrm{n}_i - \mathrm{N}r})^s\tilde{\mathrm{a}}^i_{\mathrm{N}r} \qquad r \in \frac{1}{\mathrm{N}}\mathbb{Z}, \\
&= \delta^{[\mathrm{N}]}_{\mathrm{n}_i - \mathrm{N}r,0}\,\tilde{\mathrm{a}}^i_{\mathrm{N}r}.
\end{aligned}
$$

where

$$
\delta^{[\mathrm{N}]}_{\mathrm{n},0} \equiv \sum_{\mathrm{m}\in\mathrm{Z}}\delta_{\mathrm{n},\mathrm{N}m} = \begin{cases} 1 & \mathrm{n} = 0 \ (\mathrm{mod}\ \mathrm{N}), \\ 0 & \mathrm{n} \neq 0 \ (\mathrm{mod}\ \mathrm{N}). \end{cases}
$$

Therefore the labeling r on the $\tilde{\mathrm{c}}$ oscillators is dependent on i. That is for a given one–dimensional complex subspace with eigenvalue ω^{n_i} the associated oscillators are $\tilde{\mathrm{c}}^i_r$ where $r \in \mathbb{Z} + \frac{\mathrm{n}_i}{\mathrm{N}}$. With respect to this basis

$$
\tilde{\mathrm{P}}^i_\sigma(\mathrm{z}) = \sum_{r\in\mathrm{Z}+\frac{\mathrm{n}_i}{\mathrm{N}}}\tilde{\mathrm{c}}^i_r\mathrm{z}^{-r}.
$$

Notice that $\tilde{\alpha}^{i\,*} = \tilde{\alpha}^{i+1}$, $\tilde{\mathrm{a}}^{i\,\dagger}_{\mathrm{n}} = \tilde{\mathrm{a}}^{i+1}_{-\mathrm{n}}$ and $\tilde{\mathrm{c}}^{i\,\dagger}_r = \tilde{\mathrm{c}}^{i+1}_{-r}$ for $i = \mathrm{M},\ \mathrm{M}+2, \ldots\ldots, \mathrm{rank}\,\mathrm{g}$.

Let us define the **twisted Virasoro field** to be given by,

$$
\mathrm{L}(\mathrm{z}) = \sum_{\mathrm{n}\in\mathrm{Z}}\mathrm{L}_{\mathrm{n}}\mathrm{z}^{-\frac{\mathrm{n}}{\mathrm{N}}} \equiv \frac{1}{2} : \mathrm{P}^2_\sigma(\mathrm{z}) : \ .
$$

Then the moments for each of the one–dimensional invariant subspaces are given by,

$$
\mathrm{L}^i_{\mathrm{n}} = \frac{1}{2}\sum_{r\in\mathrm{Z}} : \tilde{\mathrm{c}}^i_r\tilde{\mathrm{c}}^i_{\mathrm{n}-r} : \ .
$$

Those for the one–dimensional subspaces with eigenvalue -1 are,

$$L^i_n = \frac{1}{2} \sum_{r \in Z + \frac{1}{2}} : \tilde{c}^i_r \tilde{c}^i_{n-r} : .$$

Finally for the two–dimensional subspaces with complex eigenvalues ω^{n_i}, ω^{-n_i} we have one contribution,

$$L^i_n = \frac{1}{2} \sum_{r \in Z + \frac{n_i}{N}} : \tilde{c}^i_r \tilde{c}^{i+1}_{n-r} : .$$

The total moment, L_n, is obtained by summing the contributions from all the subspaces.

The zero moment is

$$L_0 = \frac{1}{2} p_0^2 + \sum_{\substack{i=1 \\ n_i = 0}}^{d} \sum_{\substack{n \in Z \\ n > 0}} \tilde{c}^i_{-n} \tilde{c}^i_n + \eta + (non-integer\ graded\ component) .$$

where the ambiguity due to normal ordering, η, is chosen so that

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1} .$$

It turns out that that we require

$$\eta = \frac{1}{4N^2} \sum_{i=1}^{\mathrm{rank}\, g} n_i (N - n_i) .$$

η corresponds to the **conformal weight of the vacuum** as

$$L_0 \mid 0 \rangle = \eta \mid 0 \rangle .$$

With the gradation $d = L_0$ we have the following partition function for the twisted Hilbert space H^σ,

$$P_\sigma(q) = c_\sigma \frac{\sum_{\alpha \in P_0 \Lambda_R} q^{\frac{1}{2}\alpha^2 + \eta}}{\prod_{n=1}^{\infty} (1 - q^{\frac{n}{N}})^{d(n \bmod N)}} ,$$

where c_σ is the degeneracy of the vacuum and $d(n \bmod N)$ is the dimension of the eigenspace with the eigenvalue ω^n.

To see that we have made an appropriate choice for the twisted momentum fields let us work out the following commutator, (we take $P_n = \frac{1}{N}\sum_{x=0}^{N-1}\omega^{-nx}\sigma^x$ to be the projection onto the eigenspace V_n of V as usual);

$$
\begin{aligned}
[\alpha.b_m, \beta.b_n] &= \frac{1}{N^2}\sum_{r,s=0}^{N-1}\omega^{-(mr+ns)}\sigma^r(\alpha).\sigma^s(\beta)m\delta_{m+n,0}\ ,\\
&= \frac{m}{N}\delta_{m+n,0}\sum_{x=0}^{N-1}\omega^{-mx}\sigma^x(\alpha).\beta\ ,\\
&= \frac{m}{N}\delta_{m+n,0}P_m(\alpha).\beta\ .
\end{aligned}
$$

(*i.e.* $[\alpha.c_r, \beta.c_s] = r\delta_{r+s,0}P_{Nr}(\alpha).\beta$)

This is equivalent to (3.27) of Chapter 3 (p 67) with $\alpha(m) = \alpha.b_m$. So the moments of $\alpha.P_\sigma(z)$ do indeed satisfy a $\widehat{U(1)}^{\mathrm{rank}\,g}$ Kac–Moody algebra with the gradation induced by σ,

$$
\alpha.P_\sigma(z) = \sum_{n\in Z}\alpha(n)z^{-\frac{n}{N}} \qquad \text{where } \alpha(n) \in L\left(U(1)^{\mathrm{rank}\,g},\sigma\right).
$$

The zero–mode oscillators b_0^i are again identified as **momentum operators**,

$$
p_0^i \equiv b_0^i = P_0{}^i{}_j p^j.
$$

The p^j are just the usual momentum operators. We introduce a conjugate **position operator** given by,

$$
q_0^i \equiv P_0{}^i{}_j q^j.
$$

Thus the canonical commutation relations are

$$
[q_0^i, p_0^j] = iP_0{}^{ij} \qquad (\text{as } P_0^2 = P_0).
$$

This choice is made so that $\alpha.q_0 = \alpha_0.q$ and $\alpha.p_0 = \alpha_0.p$, where $\alpha_0 \equiv P_0(\alpha)$ and p and q are the usual position and momentum operators. Hence

$$[\alpha.\mathrm{q}_0, \beta.\mathrm{p}_0] = [\alpha_0.\mathrm{q}, \beta_0.\mathrm{p}] = i\alpha_0.\beta_0 \,. \tag{6.14}$$

We now set

$$\mathrm{P}_\sigma(\mathrm{z}) \equiv i\mathrm{z}\frac{\mathrm{dQ}_\sigma}{\mathrm{dz}} \,,$$

where $\mathrm{Q}_\sigma(\mathrm{z})$ is the **twisted Fubini–Veneziano field**. Integrating and replacing the constant of integration by q_0, the position operator conjugate to p_0, we find

$$\begin{aligned}
\mathrm{Q}^i_\sigma(\mathrm{z}) &= \mathrm{q}^i_0 - i\mathrm{p}^i_0 \ln \mathrm{z}^{\frac{1}{\mathrm{N}}} + i\mathrm{N}\sum_{\mathrm{n}\in\mathrm{Z}} \frac{\mathrm{b}^i_{\mathrm{n}}}{\mathrm{n}}\mathrm{z}^{-\frac{\mathrm{n}}{\mathrm{N}}}, \\
&= \mathrm{q}^i_0 - i\mathrm{p}^i_0 \ln \mathrm{z}^{\frac{1}{\mathrm{N}}} + i\sum_{r\in\frac{1}{\mathrm{N}}\mathrm{Z}} \frac{\mathrm{c}^i_r}{r}\mathrm{z}^{-r}.
\end{aligned}$$

and

$$\tilde{\mathrm{Q}}^i_\sigma(\mathrm{z}) = (\mathrm{q}^i - i\mathrm{p}^i \ln \mathrm{z}^{\frac{1}{\mathrm{N}}})\delta^{[\mathrm{N}]}_{\mathrm{n}_i,0} + i\sum_{r\in\mathrm{Z}+\frac{\mathrm{n}_i}{\mathrm{N}}} \frac{\tilde{\mathrm{c}}^i_r}{r}\mathrm{z}^{-r}.$$

Thus

$$\alpha.\mathrm{Q}_\sigma(\mathrm{z}) = \alpha_0.\mathrm{q} - i\alpha_0.\mathrm{p}\ln \mathrm{z}^{\frac{1}{\mathrm{N}}} + i\sum_{r\in\frac{1}{\mathrm{N}}\mathrm{Z}} \frac{\alpha.\mathrm{c}_r}{r}\mathrm{z}^{-r}.$$

We define

$$\begin{aligned}
\mathrm{Q}^{>}_\sigma(\mathrm{z}) &\equiv i\sum_{r>0}\frac{\mathrm{c}_r}{r}\mathrm{z}^{-r}, \\
\mathrm{Q}^{<}_\sigma(\mathrm{z}) &\equiv i\sum_{r<0}\frac{\mathrm{c}_r}{r}\mathrm{z}^{-r}, \\
\mathrm{Q}^{<>}_\sigma(\mathrm{z}) &\equiv \mathrm{Q}^{<}_\sigma(\mathrm{z}) + \mathrm{Q}^{>}_\sigma(\mathrm{z}), \\
\mathrm{Q}^{0}_\sigma(\mathrm{z}) &\equiv \mathrm{q}_0 - i\mathrm{p}_0 \ln \mathrm{z}^{\frac{1}{\mathrm{N}}}.
\end{aligned}$$

We have the following commutation relation,

$$
\begin{aligned}
[i\alpha.\mathrm{Q}_\sigma^<(\mathrm{z}), i\beta.\mathrm{Q}_\sigma^>(\zeta)] &= -\sum_{\substack{r>0\\ s<0}} \frac{1}{rs}[\alpha.\mathrm{c}_r, \beta.\mathrm{c}_s]\mathrm{z}^{-r}\zeta^{-s}, \\
&= -\sum_{r>0} \frac{1}{r}\mathrm{P}_{\mathrm{N}r}(\alpha).\beta\left(\frac{\zeta}{\mathrm{z}}\right)^r, \\
&= -\sum_{r>0} \frac{1}{r}\left(\frac{1}{\mathrm{N}}\sum_{x=0}^{\mathrm{N}-1}\omega^{-\mathrm{N}rx}\sigma^x(\alpha).\beta\right)\left(\frac{\zeta}{\mathrm{z}}\right)^{\frac{\mathrm{N}r}{\mathrm{N}}}, \\
&= \sum_{x=0}^{\mathrm{N}-1}\sigma^x(\alpha).\beta.\left(-\sum_{\substack{\mathrm{n}\in\mathrm{Z}\\ \mathrm{n}>0}}\frac{1}{\mathrm{n}}\left(\omega^{-x}\left(\frac{\zeta}{\mathrm{z}}\right)^{\frac{1}{\mathrm{N}}}\right)^{\mathrm{n}}\right), \\
&= \sum_{x=0}^{\mathrm{N}-1}\ln\left(1-\omega^{-x}\left(\frac{\zeta}{\mathrm{z}}\right)^{\frac{1}{\mathrm{N}}}\right)^{\sigma^x(\alpha).\beta} \qquad |\zeta|<|\mathrm{z}|, \\
&= \ln \mathrm{z}^{-\alpha_0.\beta} + \sum_{x=0}^{\mathrm{N}-1}\ln(\mathrm{z}^{\frac{1}{\mathrm{N}}} - \omega^{-x}\zeta^{\frac{1}{\mathrm{N}}})^{\sigma^x(\alpha).\beta} \qquad |\zeta|<|\mathrm{z}|.
\end{aligned}
$$

Thus if we introduce the usual normal ordering : : then we have

$$
\begin{aligned}
:\mathrm{e}^{i\alpha.\mathrm{Q}_\sigma^{<>}(\mathrm{z})}::\mathrm{e}^{i\beta.\mathrm{Q}_\sigma^{<>}(\zeta)}: &=: \mathrm{e}^{i(\alpha.\mathrm{Q}_\sigma^{<>}(\mathrm{z})+\beta.\mathrm{Q}_\sigma^{<>}(\zeta))}: \mathrm{z}^{-\alpha_0.\beta} \\
&\quad\times\prod_{x=0}^{\mathrm{N}-1}(\mathrm{z}^{\frac{1}{\mathrm{N}}}-\omega^{-x}\zeta^{\frac{1}{\mathrm{N}}})^{\sigma^x(\alpha).\beta.\beta} \qquad |\zeta|<|\mathrm{z}|, \\
:\mathrm{e}^{i\alpha.\mathrm{Q}_\sigma^0(\mathrm{z})}: &= \mathrm{e}^{i\alpha_0.\mathrm{q}}\mathrm{z}^{\alpha_0.\mathrm{p}}.
\end{aligned}
$$

Also using (6.14) we have

$$
\mathrm{z}^{\alpha_0.\mathrm{p}}\mathrm{e}^{i\beta_0.\mathrm{q}} = \mathrm{e}^{i\beta_0.\mathrm{q}}\mathrm{z}^{\alpha_0.\mathrm{p}+\alpha_0.\beta_0}.
$$

We shall now give the form of the general **twisted vertex operator** and then show that this is indeed the correct form to give a representation of a Kac–Moody algebra with the gradation induced by σ.

$$
\begin{aligned}
\mathrm{V}_\sigma(\alpha,\mathrm{z}) &\equiv \mathrm{N}^{-\frac{\alpha^2}{2}}\mathrm{z}^{-\frac{(\alpha^2-\alpha_0^2)}{2}}:\mathrm{e}^{\mathrm{i}\alpha.\mathrm{Q}_\sigma(\mathrm{z})}:\mathrm{C}_\alpha, \\
&= \mathrm{N}^{-\frac{\alpha^2}{2}}\mathrm{z}^{-\frac{(\alpha^2-\alpha_0^2)}{2}}:\mathrm{e}^{\mathrm{i}\alpha.\mathrm{Q}_\sigma^{<>}(\mathrm{z})}:\mathrm{e}^{i\alpha_0.\mathrm{q}}\mathrm{C}_\alpha\mathrm{z}^{\alpha_0.\mathrm{p}}.
\end{aligned}
$$

where C_α is a 2–cocycle operator acting on the zero–mode space. If we define $\hat{\mathrm{C}}(\alpha) \equiv \mathrm{e}^{i\alpha_0.\mathrm{q}}\mathrm{C}_\alpha$ then in addition we require that $\hat{\mathrm{C}}(\alpha)$ satisfies

1.

$$[\alpha.\mathrm{p}, \hat{\mathrm{C}}(\beta)] = \alpha.\beta\hat{\mathrm{C}}(\beta)\,. \tag{6.15}$$

Thus in particular $\mathrm{z}^{\alpha\cdot\mathrm{p}}\hat{\mathrm{C}}(\beta) = \hat{\mathrm{C}}(\beta)\mathrm{z}^{\alpha\cdot\mathrm{p}+\alpha.\beta}$

2.

$$\hat{\mathrm{C}}(\alpha)\hat{\mathrm{C}}(\beta) = \varepsilon_{\mathrm{c}}(\alpha,\beta)\hat{\mathrm{C}}(\alpha+\beta) \tag{6.16}$$

where $\varepsilon_{\mathrm{C}} : \Lambda_R \times \Lambda_R \rightarrow \mathbb{C} - \{0\}$ is a normalised 2–cocycle associated with C satisfying

$$\frac{\varepsilon_{\mathrm{c}}(\alpha,\beta)}{\varepsilon_{\mathrm{c}}(\beta,\alpha)} = \mathrm{C}(\alpha,\beta) \equiv \prod_{r=0}^{\mathrm{N}-1}(-\omega^r)^{\sigma^r(\alpha).\beta}\,.$$

3.

$$\begin{aligned}\psi_\alpha\hat{\mathrm{C}}(\sigma(\alpha)) &= \omega^{-\tilde{\alpha}.\mathrm{p}+\frac{\tilde{\alpha}.\alpha}{2}}\hat{\mathrm{C}}(\alpha)\,, \\ &= \hat{\mathrm{C}}(\alpha)\omega^{-\tilde{\alpha}.\mathrm{p}-\frac{\tilde{\alpha}.\alpha}{2}} \quad \text{using (6.15).}\end{aligned} \tag{6.17}$$

where $\psi_\alpha : \Lambda_R \rightarrow \mathbb{C} - \{0\}$ with

$$\begin{aligned}(i) \quad \psi_\alpha\psi_\beta &= \frac{\varepsilon(\alpha,\beta)}{\varepsilon(\sigma(\alpha),\sigma(\beta))}\psi_{\alpha+\beta} \\ (ii) \quad \psi_\alpha\psi_{-\alpha} &= 1 \qquad\qquad\qquad \text{(cf Chapter 3 p 58)} \\ (iii) \quad \psi^*_\alpha &= \psi_\alpha\end{aligned}$$

ψ_α is the phase associated with the extension of σ to a Lie algebra automorphism. If we define $\eta(x,\alpha) \equiv \psi_\alpha\psi_{\sigma(\alpha)}.....\psi_{\sigma^x(\alpha)}$ then (6.17) is equivalent to

$$\eta(x,\alpha)\hat{\mathrm{C}}\left(\sigma^x(\alpha)\right) = \omega^{-x\tilde{\alpha}.\mathrm{p}+x\frac{\tilde{\alpha}.\alpha}{2}}\hat{\mathrm{C}}(\alpha)\,.$$

We shall show how to construct such a set of operators in the next section.

Choice of 2 – cocycle.

As $C(\alpha,\beta)$ is bilinear we can make the further restriction that ε_c is bilinear. This will be useful in later calculations. In this case the 2–cocycle condition is replaced by,

$$\varepsilon_c(\alpha+\beta,\gamma) = \varepsilon_c(\alpha,\gamma)\varepsilon_c(\beta,\gamma)\,.$$

This also means that we have

1. $\varepsilon_c(\alpha,0) = 1$,
2. $\varepsilon_c(\alpha,\beta) = \frac{1}{\varepsilon_c(\alpha,-\beta)}$.

With this choice we can explicitly construct a set of $\varepsilon_c(\alpha,\beta)$ by generalising the method of [2] given in Chapter 1. Choose

$$\varepsilon_c(\alpha_i,\alpha_j) = \begin{cases} 1 & i \geq j\,, \\ C(\alpha_i,\alpha_j) & i < j\,. \end{cases}$$

We can expand $V_\sigma(\alpha,z)$ as follows

$$V_\sigma(\alpha,z) = \sum_{n\in\frac{1}{N}Z} V_n^\sigma(\alpha) z^{-n-\frac{\alpha^2}{2}}$$

where the moments are given by

$$V_n^\sigma(\alpha) \equiv \frac{1}{2\pi i}\oint \frac{dz}{z} z^{N\left(n+\frac{\alpha^2}{2}\right)} V_\sigma(\alpha,z^N)\,.$$

The z contour encircles the origin once positively. Now

$$\begin{aligned} V_\sigma(\alpha,z)V_\sigma(\beta,\zeta) =& N^{-\frac{(\alpha+\beta)^2}{2}} z^{-\frac{(\alpha^2-\alpha_0^2)}{2}} \zeta^{-\frac{(\beta^2-\beta_0^2)}{2}} : e^{i(\alpha.Q_\sigma(z)+\beta.Q_\sigma(\zeta))} : C_{\alpha+\beta} \\ & N^{\alpha.\beta}\varepsilon_c(\alpha,\beta) \prod_{y=0}^{N-1} (z^{\frac{1}{N}} - \omega^{-y}\zeta^{\frac{1}{N}})^{\sigma^y(\alpha).\beta} \qquad |\,\zeta\,|<|\,z\,|\,. \end{aligned} \tag{6.18}$$

Let us therefore define,

$$U_{\alpha,\beta}(z,\zeta) \equiv N^{-\frac{(\alpha+\beta)^2}{2}} z^{-\frac{(\alpha^2-\alpha_0^2)}{2}} \zeta^{-\frac{(\beta^2-\beta_0^2)}{2}} : e^{i(\alpha.Q_\sigma(z)+\beta.Q_\sigma(\zeta))} : C_{\alpha+\beta}\,.$$

Then we can rewrite (6.18) as,

$$\mathrm{V}_\sigma(\alpha,z)\mathrm{V}_\sigma(\beta,\zeta) = \mathrm{U}_{\alpha,\beta}(z,\zeta)\mathrm{N}^{\alpha.\beta}\varepsilon_c(\alpha,\beta)\prod_{y=0}^{N-1}(z^{\frac{1}{N}} - \omega^{-y}\zeta^{\frac{1}{N}})^{\sigma^y(\alpha).\beta} \qquad |\,\zeta\,|<|\,z\,|.$$

Similiarly we have

$$\mathrm{V}_\sigma(\beta,\zeta)\mathrm{V}_\sigma(\alpha,z) = \mathrm{U}_{\alpha,\beta}(z,\zeta)\mathrm{N}^{\alpha.\beta}\varepsilon_c(\beta,\alpha)\prod_{y=0}^{N-1}(\zeta^{\frac{1}{N}} - \omega^{-y}z^{\frac{1}{N}})^{\sigma^y(\beta).\alpha} \qquad |\,z\,|<|\,\zeta\,|.$$

But

$$\begin{aligned}\varepsilon_c(\beta,\alpha) &= \varepsilon_c(\alpha,\beta)C(\beta,\alpha),\\ &= \varepsilon_c(\alpha,\beta)\prod_{r=0}^{N-1}(-\omega^r)^{\sigma^r(\beta).\alpha},\end{aligned}$$

hence

$$\begin{aligned}\varepsilon_c(\beta,\alpha)\prod_{y=0}^{N-1}(\zeta-\omega^{-y}z)^{\sigma^y(\beta).\alpha} &= \varepsilon_c(\alpha,\beta)\prod_{y=0}^{N-1}(-\omega^y\zeta+z)^{\sigma^y(\beta).\alpha},\\ &= \varepsilon_c(\alpha,\beta)\prod_{y=0}^{N-1}(z-\omega^{-y}\zeta)^{\sigma^y(\alpha).\beta}.\end{aligned}$$

Thus $\mathrm{V}_\sigma(\alpha,z^N)\mathrm{V}_\sigma(\beta,\zeta^N)$" = "$\mathrm{V}_\sigma(\beta,\zeta^N)\mathrm{V}_\sigma(\alpha,z^N)$ where the equality is in the forms of the two sides as the lefthand side is strictly only defined for $|\,\zeta\,|<|\,z\,|$ and the righthand side for $|\,z\,|<|\,\zeta\,|$. This allows us to use the usual contour rearrangement argument when we work out the commutation relations of the moments.

$$\begin{aligned}&[\mathrm{V}^\sigma_m(\alpha),\mathrm{V}^\sigma_n(\beta)] =\\ &\quad\frac{1}{(2\pi i)^2}\left\{\oint_{|z|>|\zeta|}\frac{dz}{z}z^{N\left(m+\frac{\alpha^2}{2}\right)}\oint\frac{d\zeta}{\zeta}\zeta^{N\left(n+\frac{\beta^2}{2}\right)} - \oint_{|z|<|\zeta|}\frac{dz}{z}z^{N\left(m+\frac{\alpha^2}{2}\right)}\oint\frac{d\zeta}{\zeta}\zeta^{N\left(n+\frac{\beta^2}{2}\right)}\right\}\\ &\quad\times\ \mathrm{U}_{\alpha,\beta}(z^N,\zeta^N)\mathrm{N}^{\alpha.\beta}\varepsilon_c(\alpha,\beta)\prod_{y=0}^{N-1}(z-\omega^{-y}\zeta)^{\sigma^y(\alpha).\beta},\\ &\quad= \mathrm{N}^{\alpha.\beta}\varepsilon_c(\alpha,\beta)\frac{1}{2\pi i}\oint_0\frac{d\zeta}{\zeta}\zeta^{N\left(n+\frac{\beta^2}{2}\right)}\mathrm{I}_{\alpha,\beta}(\zeta),\end{aligned}$$

where

$$\mathrm{I}_{\alpha,\beta}(\zeta) \equiv \frac{1}{2\pi i} \oint_{\{\omega^{-x}\zeta\}} \frac{\mathrm{d}z}{z} z^{\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}\right)} \mathrm{U}_{\alpha,\beta}(z^{\mathrm{N}}, \zeta^{\mathrm{N}}) \prod_{y=0}^{\mathrm{N}-1} (z - \omega^{-y}\zeta)^{\sigma^y(\alpha).\beta}$$

and $\{\omega^{-x}\zeta\}$ denotes a contour which encircles all the points ω^{-x} with $\sigma^x(\alpha).\beta < 0$, which correspond to poles of the integrand, but excludes the origin. The ζ contour encircles the origin positively once.

$$\mathrm{I}_{\alpha,\beta}(\zeta) = \sum_{x=0}^{\mathrm{N}-1} \mathrm{I}^x_{\alpha,\beta}(\zeta)$$

where

$$\begin{aligned}
\mathrm{I}^x_{\alpha,\beta}(\zeta) &\equiv \mathrm{Res}\left(z^{\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}\right)-1} \mathrm{U}_{\alpha,\beta}(z^{\mathrm{N}}, \zeta^{\mathrm{N}}) \prod_{y=0}^{\mathrm{N}-1} (z - \omega^{-y}\zeta)^{\sigma^y(\alpha).\beta} \right)\Bigg|_{z=\omega^{-x}\zeta}, \\
&= \begin{cases} 0 & \sigma^x(\alpha).\beta \geq 0, \\ \frac{1}{(\mathrm{M}-1)!} \frac{\mathrm{d}^{\mathrm{M}-1}}{\mathrm{d}z^{\mathrm{M}-1}} \Big(z^{\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}\right)-1} \mathrm{U}_{\alpha,\beta}(z^{\mathrm{N}}, \zeta^{\mathrm{N}}) & \\ \times \displaystyle\prod_{\substack{y=0 \\ y \neq x}}^{\mathrm{N}-1} (z - \omega^{-y}\zeta)^{\sigma^y(\alpha).\beta} \Big)\Big|_{z=\omega^{-x}\zeta} & \sigma^x(\alpha).\beta = -\mathrm{M} < 0. \end{cases}
\end{aligned}$$

We shall now look at the commutator of two twisted vertex operators defined on the roots of length squared two of a simply laced algebra *i.e.* $\alpha,\ \beta \in \Phi_{\mathrm{g}}$ and $\alpha^2 = \beta^2 = 2$. Fortunately in this case we only have the possibilities $\mathrm{M} =$ 1, 2.

<u>M = 1 :</u>

$$\mathrm{I}^x_{\alpha,\beta}(\zeta) = (\omega^{-x}\zeta)^{\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}+\tilde{\alpha}.\beta\right)-1+1} \mathrm{U}_{\alpha,\beta}\left((\omega^{-x}\zeta)^{\mathrm{N}}, \zeta^{\mathrm{N}}\right) \prod_{\substack{y=0 \\ y \neq x}}^{\mathrm{N}-1} (1 - \omega^{x-y})^{\sigma^y(\alpha).\beta}.$$

Firstly notice that,

$$\begin{aligned}
\alpha.\mathrm{Q}_\sigma^{<>}(\omega^{-x}\zeta) &= i\sum_{\mathrm{n}\neq 0}\sum_{r=0}^{\mathrm{N}-1}\omega^{-\mathrm{n}(r-x)}\sigma^r(\alpha).\frac{\mathrm{a_n}}{\mathrm{n}}\zeta^{-\mathrm{n}}\,, \\
&= i\sum_{\mathrm{n}\neq 0}\sum_{s=0}^{\mathrm{N}-1}\omega^{-\mathrm{n}s}\sigma^{s+x}(\alpha).\frac{\mathrm{a_n}}{\mathrm{n}}\zeta^{-\mathrm{n}}\,, \\
&= \sigma^x(\alpha).\mathrm{Q}_\sigma^{<>}(\zeta)\,.
\end{aligned}$$

Therefore

$$\alpha.\mathrm{Q}_\sigma^{<>}(\omega^{-x}\zeta)+\beta.\mathrm{Q}_\sigma^{<>}(\zeta) = \left(\sigma^x(\alpha)+\beta\right).\mathrm{Q}_\sigma^{<>}(\zeta)\,.$$

Thus

$$\begin{aligned}
\mathrm{U}_{\alpha,\beta}\left((\omega^{-x}\zeta)^{\mathrm{N}},\zeta^{\mathrm{N}}\right) &= \mathrm{N}^{-\frac{(\alpha+\beta)^2}{2}} : \mathrm{e}^{\mathrm{i}(\sigma^x(\alpha)+\beta).\mathrm{Q}_\sigma^{<>}(\zeta^{\mathrm{N}})} : \mathrm{e}^{(\alpha+\beta)_0.\mathrm{q}}\mathrm{C}_{\alpha+\beta} \\
&\quad\times\ \zeta^{\mathrm{N}(\alpha+\beta)_0.\mathrm{p}}\zeta^{\frac{-\mathrm{N}((\alpha^2+\beta^2-(\alpha_0^2+\beta_0^2))}{2}}(\omega^{-x})^{\mathrm{N}\alpha_0.\mathrm{p}}(\omega^{-x})^{\frac{-\mathrm{N}(\alpha^2-\alpha_0^2)}{2}} \\
&= \mathrm{V}_\sigma\left(\sigma^x(\alpha)+\beta,\zeta^{\mathrm{N}}\right)\mathrm{N}^{-\alpha.\beta+\sigma^x(\alpha).\beta}\eta(x,\alpha)\frac{\varepsilon_{\mathrm{c}}(\sigma^x(\alpha),\beta)}{\varepsilon_{\mathrm{c}}(\alpha,\beta)} \\
&\quad\times\ \zeta^{\mathrm{N}(\sigma^x(\alpha).\beta-\alpha_0.\beta_0)}.
\end{aligned} \tag{6.19}$$

As

$$\begin{aligned}
\mathrm{C}_{\alpha+\beta} &= \frac{1}{\varepsilon_{\mathrm{c}}(\alpha,\beta)}\mathrm{C}_\alpha\mathrm{C}_\beta\,, \\
&= \frac{1}{\varepsilon_{\mathrm{c}}(\alpha,\beta)}\eta(x,\alpha)\omega^{x\tilde{\alpha}.\mathrm{p}}\omega^{x\frac{\tilde{\alpha}.\alpha}{2}}\mathrm{C}_{\sigma^x(\alpha)}\mathrm{C}_\beta\,, \\
&= \frac{\varepsilon_{\mathrm{c}}(\sigma^x(\alpha),\beta)}{\varepsilon_{\mathrm{c}}(\alpha,\beta)}\eta(x,\alpha)\omega^{x\mathrm{N}\alpha_0.\mathrm{p}}\omega^{x\mathrm{N}\frac{\alpha_0^2}{2}}\mathrm{C}_{\sigma^x(\alpha)+\beta}\,.
\end{aligned}$$

If we now define

$$\varepsilon'(\alpha,\beta) \equiv \prod_{s=1}^{\mathrm{N}-1}(1-\omega^{-s})^{\sigma^s(\alpha).\beta}$$

then we have,

$$\begin{aligned}
\zeta^{\mathrm{N}\left(\mathrm{n}+\frac{\beta^2}{2}\right)}\mathrm{I}^x_{\alpha,\beta}(\zeta) &= \omega^{-\mathrm{N}mx}\zeta^{\mathrm{N}((m+n)+\frac{(\sigma^x(\alpha)+\beta)^2}{2}))}\mathrm{N}^{-\alpha.\beta+\sigma^x(\alpha).\beta} \\
&\quad\times\ \eta(x,\alpha)\frac{\varepsilon_{\mathrm{c}}(\sigma^x(\alpha),\beta)}{\varepsilon_{\mathrm{c}}(\alpha,\beta)}\varepsilon'\left(\sigma^x(\alpha),\beta\right)\mathrm{V}_\sigma\left(\sigma^x(\alpha)+\beta,\zeta^{\mathrm{N}}\right).
\end{aligned}$$

In addition we know that $\sigma^x(\alpha).\beta = -1$, and therefore $(\sigma^x(\alpha)+\beta)^2 = 2$.

M = 2 :

$$\mathrm{I}^{x}_{\alpha,\beta}(\zeta) = \frac{\mathrm{d}}{\mathrm{dz}}\left(\mathrm{z}^{\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}\right)-1}\mathrm{U}_{\alpha,\beta}(\mathrm{z}^{\mathrm{N}},\zeta^{\mathrm{N}})\prod_{\substack{y=0\\y\neq x}}^{\mathrm{N}-1}(\mathrm{z}-\omega^{-y}\zeta)^{\sigma^{y}(\alpha).\beta}\right)\Bigg|_{\mathrm{z}=\omega^{-x}\zeta},$$

$$= \left\{\left(\left(\mathrm{N}\left(\mathrm{m}+\frac{\alpha^2}{2}\right)-1\right)+\mathrm{z}\sum_{\substack{w\neq 0\\w=x}}^{\mathrm{N}-1}\frac{\sigma^{w}(\alpha).\beta}{(\mathrm{z}-\omega^{-w}\zeta)}\right)\mathrm{U}_{\alpha,\beta}(\mathrm{z}^{\mathrm{N}},\zeta^{\mathrm{N}})\right.$$

$$\left.+\,\mathrm{z}\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{U}_{\alpha,\beta}(\mathrm{z}^{\mathrm{N}},\zeta^{\mathrm{N}})\right\}\mathrm{z}^{\mathrm{N}(\mathrm{m}+\frac{\alpha^2}{2})-2}\prod_{\substack{y=0\\y\neq x}}^{\mathrm{N}-1}(\mathrm{z}-\omega^{-y}\zeta)^{\sigma^{y}(\alpha).\beta}\Bigg|_{\mathrm{z}=\omega^{-x}\zeta}.$$

Now $\sigma^{x}(\alpha).\beta = -2$ implies that $\sigma^{x}(\alpha) = -\beta$ therefore,

$\underline{(i)}$
$$\sum_{\substack{w=0\\w\neq x}}^{\mathrm{N}-1}\frac{\sigma^{w}(\alpha).\beta}{(\omega^{-x}-\omega^{-w})} = \frac{\beta^2-\tilde{\beta}.\beta}{2\omega^{-x}}.$$

$\underline{(ii)}$
$$\mathrm{z}\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{U}_{\alpha,\beta}(\mathrm{z}^{\mathrm{N}},\zeta^{\mathrm{N}})\Big|_{\mathrm{z}=\omega^{-x}\zeta} = \left\{\mathrm{N}\frac{(\alpha_0^2-\alpha^2)}{2}-\mathrm{N}\beta.\mathrm{P}_{\sigma}(\zeta^{\mathrm{N}})\right\}\mathrm{U}_{\alpha,\beta}\left((\omega^{-x}\zeta)^{\mathrm{N}},\zeta^{\mathrm{N}}\right).$$

$\underline{(iii)}$
$$\mathrm{U}_{\alpha,\beta}\left((\omega^{-x}\zeta)^{\mathrm{N}},\zeta^{\mathrm{N}}\right) = \mathrm{C}_0\mathrm{N}^{-\alpha.\beta-2}\eta(x,\alpha)\frac{\varepsilon_{\mathrm{C}}(-\beta,\beta)}{\varepsilon_{\mathrm{C}}(\alpha,\beta)}\zeta^{-\mathrm{N}(\beta^2+\alpha_0.\beta_0)}.$$

$\underline{(iv)}$
$$\prod_{\substack{y=0\\y\neq x}}^{\mathrm{N}-1}(\mathrm{z}-\omega^{-y}\zeta)^{\sigma^{y}(\alpha).\beta}\Big|_{\mathrm{z}=\omega^{-x}\zeta} = \zeta^{\mathrm{N}\tilde{\alpha}.\beta+2}\varepsilon'(\sigma^{x}(\alpha),\beta)\omega^{-2x}.$$

Proof : $\underline{(i)}$

$$\mathrm{LHS} = \frac{1}{\omega^{-x}}\sum_{s=1}^{\mathrm{N}-1}\frac{\sigma^{s-x}(\alpha).\beta}{(1-\omega^{s})},$$

$$= -\frac{1}{\omega^{-x}}\sum_{s=1}^{\mathrm{N}-1}\frac{\sigma^{s}(\beta).\beta}{(1-\omega^{s})},$$

$$= -\frac{1}{\omega^{-x}}\begin{cases}\sum_{s=1}^{\frac{(\mathrm{N}-2)}{2}}\sigma^{s}(\beta).\beta+\frac{\sigma^{\frac{\mathrm{N}}{2}}(\beta).\beta}{2} & \mathrm{N}\in 2\mathbb{Z},\\ \sum_{s=1}^{\frac{(\mathrm{N}-1)}{2}}\sigma^{s}(\beta).\beta & \mathrm{N}\in 2\mathbb{Z}+1,\end{cases}$$

$$= \mathrm{RHS}.$$

(ii)

$$\frac{d}{dz}U_{\alpha,\beta}(z^N,\zeta^N)\bigg|_{z=\omega^{-x}\zeta} = \Bigg\{N\frac{(\alpha_0^2-\alpha^2)}{2}z^{-1}U_{\alpha,\beta}(z^N,\zeta^N)+N^{-\frac{(\alpha+\beta)^2}{2}}z^{-N\frac{(\alpha^2-\alpha_0^2)}{2}}$$

$$\times\ \zeta^{-N\frac{(\beta^2-\beta_0^2)}{2}} : i\alpha.\frac{dQ_\sigma}{dz}(z^N)e^{i(\alpha.Q_\sigma(z)+\beta.Q_\sigma(\zeta))} : C_{\alpha+\beta}\Bigg\}\bigg|_{z=\omega^{-x}\zeta}.$$

But $i\alpha.\frac{dQ_\sigma}{dz}(z^N) = \frac{N}{z}\alpha.P_\sigma(z^N)$, and

$$\begin{aligned}\alpha.P_\sigma(z^N)|_{z=\omega^{-x}\zeta} &= \sigma^x(\alpha).P_\sigma(\zeta^N)\,,\\ &= -\beta.P_\sigma(\zeta^N)\,.\end{aligned}$$

Thus we have,

$$\begin{aligned}\text{LHS} &= \left\{N\frac{(\alpha_0^2-\alpha^2)}{2}-N\beta.P_\sigma(\zeta^N)\right\}U_{\alpha,\beta}\left((\omega^{-x}\zeta)^N,\zeta^N\right).\\ &= \text{RHS}\,.\end{aligned}$$

(iii) Put $\sigma^x(\alpha).\beta = -2$, $\sigma^x(\alpha) = -\beta$ in (6.19).

(iv)

$$\begin{aligned}\text{LHS} &= (\omega^{-x}\zeta)^{N\tilde{\alpha}.\beta+2}\prod_{\substack{y=0\\y\neq x}}^{N-1}(1-\omega^{x-y})^{\sigma^y(\alpha).\beta}\,,\\ &= \text{RHS}.\end{aligned}$$

Therefore,

$$I^x_{\alpha,\beta}(\zeta) = \left\{N\left(m+\frac{\alpha^2}{2}\right)-1+1-\frac{\tilde{\beta}.\beta}{2}+N\frac{(\alpha_0^2-\alpha^2)}{2}-N\beta.P_\sigma(\zeta^N)\right\}$$

$$\times\ C_0N^{-\alpha.\beta-2}\eta(x,\alpha)\frac{\varepsilon_c(-\beta,\beta)}{\varepsilon_c(\alpha,\beta)}\varepsilon'(-\beta,\beta)\zeta^{N(m+\frac{\alpha^2}{2}-\beta^2)}\omega^{-Nmx}\,.$$

But $\sigma^x(\alpha) = -\beta$ implies that $\tilde{\alpha}.\alpha = \tilde{\beta}.\beta$, thus

$$\zeta^{N\left(n+\frac{\beta^2}{2}\right)}I^x_{\alpha,\beta}(\zeta) = \omega^{-x\tilde{\alpha}.\beta}\left\{m-\beta.P_\sigma(\zeta^N)\right\}C_0N^{-\alpha.\beta-1}\eta(x,\alpha)\frac{\varepsilon_c(-\beta,\beta)}{\varepsilon_c(\alpha,\beta)}\varepsilon'(-\beta,\beta)\zeta^{N(n+m)}\,.$$

Thus finally we have,

$$
\begin{aligned}
&[\mathrm{V}^\sigma_\mathrm{m}(\alpha),\mathrm{V}^\sigma_\mathrm{n}(\beta)] = \\
&= \mathrm{N}^{\alpha.\beta}\varepsilon_c(\alpha,\beta)\frac{1}{2\pi i}\oint_0 \frac{d\zeta}{\zeta}\zeta^{\mathrm{N}\left(\mathrm{n}+\frac{\beta^2}{2}\right)}\sum_{x=0}^{\mathrm{N}-1}\mathrm{I}^x_{\alpha,\beta}(\zeta), \\
&= \frac{1}{\mathrm{N}}\left\{\sum_{\sigma^x(\alpha).\beta=-1}\omega^{-\mathrm{N}\mathrm{m}x}\eta(x,\alpha)\varepsilon(\sigma^x(\alpha),\beta)\mathrm{V}^\sigma_{\mathrm{m}+\mathrm{n}}(\sigma^x(\alpha)+\beta\ )\right. \\
&\left.+\sum_{\sigma^x(\alpha).\beta=-2}\omega^{-\mathrm{N}\mathrm{m}x}\eta(x,\alpha)\varepsilon(-\beta,\beta)\left\{\mathrm{m}\delta_{\mathrm{m}+\mathrm{n},0}-\beta.\mathrm{c}_{\mathrm{n}+\mathrm{m}}\right\}\right\}.
\end{aligned}
$$

Where,

$$\varepsilon(\alpha,\beta) \equiv \varepsilon_c(\alpha,\beta)\varepsilon'(\alpha,\beta)$$

is a 2–cocycle associated with $\mathrm{S}(\alpha,\beta) = (-1)^{\alpha.\beta}$ by Lemma (6.1). So the moments of the twisted vertex operators defined on roots of length squared two along with the moments of the twisted momentum field and the identity operator do indeed give a representation of $\mathrm{g}^{(\tau)}$ in the gradation induced by σ (See equations (3.27) to (3.29) on p 67). Notice that to calculate a particular commutator $[\mathrm{V}^\sigma_\mathrm{m}(\alpha),\mathrm{V}^\sigma_\mathrm{n}(\beta)]$ explicitly, all that is required is the set of inner products $(\alpha.\beta,\ \sigma(\alpha).\beta,\ ...,\sigma^{\mathrm{N}-1}(\alpha).\beta)$.

6.3 The zero–mode space and cocycle operators.

We shall now proceed to examine the zero–mode Hilbert space in a little more detail and in particular give an explicit construction of the cocycle matrices [5]. The subtlety of the twisted vertex operator construction lies in in this space. One of the main results of this examination is to see that the zero–mode space can be written in the form,

$$\mathrm{V}^\sigma = \mathrm{C}(\mathrm{P}_0\Lambda_R)\otimes\overline{\mathrm{V}}^\sigma$$

where

1. $\mathrm{P}_0\Lambda_R$ is the projection of the root lattice onto the invariant subspace and $\mathrm{C}(\mathrm{P}_0\Lambda_R)$ is the Hilbert space spanned by states of the form $\mid\alpha_0\rangle,\ \alpha_0 \in \mathrm{P}_0\Lambda_R$.
2. $\overline{\mathrm{V}}^\sigma$ is the space of an irreducible projective representation of the lattice $\mathrm{N} = (1-\mathrm{P}_0)\mathrm{V}\cap\Lambda_R$ *i.e.* $\mathrm{N} = \{\alpha \in \Lambda_R \mid \alpha.\beta_0 = 0\ \ \forall\ \beta_0 \in \mathrm{P}_0\Lambda_R\} \subset \Lambda_R$.

Let us review the ordinary, untwisted zero–mode space first. This consists of the complex linear span of momentum eigenstates $|\,\alpha\rangle$, $\alpha \in \Lambda_R$, which are built up from a non-degenerate vacuum state, $|\,0\rangle$, by the zero–mode operators $\mathrm{e}^{i\alpha.\mathrm{q}}$,

$$\begin{aligned} \mathrm{e}^{i\alpha.\mathrm{q}}\,|\,0\rangle &= |\,\alpha\rangle\ , \\ \mathrm{p}\,|\,0\rangle &= 0\,|\,0\rangle\ . \end{aligned}$$

The zero–mode operators appearing in the vertex operators, which we call **cocycle operators**, are necessarily of the form,

$$\hat{\mathrm{C}}(\alpha) = \mathrm{e}^{i\alpha.\mathrm{q}}\mathrm{C}_\alpha\ ,$$

where C_α is a function of momentum only so that we still have,

$$[\alpha.\mathrm{p}, \hat{\mathrm{C}}(\beta)] = \alpha.\beta\hat{\mathrm{C}}(\beta)\,.$$

This is necessary if the vertex operators are to have the correct weights to form a representation of $\mathrm{g}^{(1)}$. Physically this means the vertex operator $\mathrm{V}(\beta,\mathrm{z})$ creates momentum β.

These operators must also satisfy,

1. $\hat{\mathrm{C}}(\alpha)\hat{\mathrm{C}}(\beta) = \varepsilon(\alpha,\beta)\hat{\mathrm{C}}(\alpha+\beta)$ where $\varepsilon : \Lambda_R\times\Lambda_R \to \{\pm1\}$ is a normalised 2–cocycle associated with $(-1)^{\alpha.\beta}$,

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = (-1)^{\alpha.\beta}\ .$$

This just means that the $\hat{\mathrm{C}}(\alpha)$ must form a projective representation of Λ_R with factor set $\varepsilon(\alpha,\beta)$. It is necessary so that the phase produced by reversing the order of two such zero–mode operators cancels the extraneous phase arising from the interchange of the non–zero moded pieces of the two vertex operators. It thus allows us to form the commutators of vertex operators.

Note: There is an alternative construction where the C_α are taken to be products of generalised γ matrices rather than functions of momentum [47]. Such matrices $\hat{\gamma}_i$ are defined on the simple roots so that

$$\hat{\gamma}_i\hat{\gamma}_j = S(\alpha_i, \alpha_j)\hat{\gamma}_j\hat{\gamma}_i .$$

A general $\hat{\gamma}_\alpha$ is then given by $\hat{\gamma}_\alpha = \hat{\gamma}_1^{n_1}.....\hat{\gamma}_{\mathrm{rank}\,g}^{n_{\mathrm{rank}\,g}}$, where $\alpha = \sum_{i=1}^{\mathrm{rank}\,g} n_i\alpha_i$. This will automatically introduce a vacuum degeneracy unless $C(\alpha, \beta)$ is trivial.

Similarly in the case of the twisted vertex operator we have,

$$\hat{C}(\alpha) = e^{i\alpha_0 . q} C_\alpha ,$$

where C_α is a function of momentum only. Thus

$$[\alpha_0 . p, \hat{C}(\beta)] = \alpha_0 . \beta\hat{C}(\beta) , \tag{6.20}$$

which is the correct commutation relation for the zero–modes of such a graded representation (see (3.31) p 67).

The representation initially appears to be acting only on the Fock space of momentum eigenstates, $| \alpha_0\rangle$, where $\alpha_0 \in P_0\Lambda_R$ and $P_0\Lambda_R$ is the projection of Λ_R onto the invariant subspace. However for $\alpha \in \Lambda_R$ with $\alpha_0 = 0$ the zero–mode of the vertex operator is given by

$$V_0^\sigma(\alpha) = C_\alpha$$

which in particular does **not** annihilate the vacuum. The set of C_α for which $\alpha_0 = 0$ must form a representation of the finite Lie subalgebra g_0 of $g^{(\tau)}$ that commutes with L_0. Hence in general the vaccuum is n–fold degenerate and the C_α form an n–dimensional representation of g_0. We write the degenerate vaccuum as

$$| 0, \phi_i\rangle \equiv | 0\rangle\otimes | \phi_i\rangle \qquad i = 1, ..., n.$$

The Fock space of this twisted vertex operator representation then consists of the complex span of states of the form $| \alpha_0, \phi_i\rangle \equiv | \alpha_0\rangle\otimes | \phi_i\rangle$ where $| \phi_i\rangle \in \underline{n}(g_0)$, an n–dimensional representation of g_0 *i.e.*

$$V^\sigma = \mathbf{C}(P_0\Lambda_R) \otimes \underline{n}(g_0).$$

Here $\mathbf{C}(P_0\Lambda_R)$ denotes the complex linear span of states $| \alpha\rangle$ such that $\alpha \in P_0\Lambda_R$. The position and momentum operators act only on the first part of such a state whilst C_α acts on both parts.

In the twisted case the $\hat{C}(\alpha)$ must satisfy (6.16) and (6.17).

This time the $\hat{C}(\alpha)$ must form a projective representation of Λ_R with a factor set $\varepsilon_c(\alpha,\beta)$ if we are to be able to obtain commutation relations. (6.17) comes about as $V_\sigma(\alpha,z)$ represents the element in the loop algebra $\hat{L}(g,\Sigma)$ corresponding to the step operator E_α *i.e.*

$$V_\sigma(\alpha,z^N) = \sum_{n\in Z} E_\alpha(n)z^{-n}.$$

It must therefore be invariant under the corresponding Kac–Moody automorphism, $\hat{\Sigma}$,

$$\hat{\Sigma}(V_\sigma(\alpha,z^N)) = V_\sigma(\alpha,z^N).$$

But,

$$\begin{aligned}\hat{\Sigma}(E_\alpha(n)) &= \omega^{-n}(\Sigma(E_\alpha))(n),\\ &= \omega^{-n}\psi_\alpha E_{\sigma(\alpha)}(n),\\ &= E_\alpha(n).\end{aligned}$$

Thus we must have,

$$\psi_\alpha V_\sigma(\sigma(\alpha),(\omega z)^N) = V_\sigma(\alpha,z^N).$$

Now,

$$\sigma(\alpha).Q_\sigma^{<>}\left((\omega z)^N\right) = \alpha.Q_\sigma^{<>}(z^N),$$

so that for the zero–mode space we must have

$$(\omega z)^{N\frac{\alpha_0^2}{2}}\psi_\alpha\hat{C}\left(\sigma(\alpha)\right)(\omega z)^{\tilde{\alpha}.p} = z^{N\frac{\alpha_0^2}{2}}\hat{C}(\alpha)z^{\tilde{\alpha}.p}$$

which gives (6.17). In fact for the untwisted case if we take the identity automorphism in (6.17) then it reduces to

$$\hat{C}(\alpha) = e^{-2\pi i\alpha.p}\hat{C}(\alpha).$$

This corresponds to the requirement that

$$V(\alpha,z) = V(\alpha,e^{2\pi i}z).$$

Thus the Hilbert space has to be invariant under the action of $e^{2\pi i\alpha.p}$, $\forall\ \alpha \in \Lambda_R$ and therefore must be a subset of $C(\Lambda_W)$, which it is. More generally we could choose the vacuum state to be $|\gamma\rangle$ where $\gamma \in \Lambda_W$. Thus we have a different representation of $\hat{g}$ for each coset of Λ_R in Λ_W. Similiarly in the twisted case we may form an alternative representation by taking the vacuum state to be $|\gamma\rangle$ where $\gamma \in P_0\Lambda_W$ such that $\gamma.\alpha_0 \in \mathbb{Z}\ \forall\alpha_0 \in P_o\Lambda_R$. This is a special case of a γ**–shifted operator** construction given in Section 10 of [5].

Before proceeding to construct a suitable set of cocycle operators we need to distinguish some important sublattices of the root lattice, Λ_R. Considered as a group under the addition of lattice vectors Λ_R and any of its subgroups are finitely generated free abelian groups. We shall write a **finitely generated group** with generators $g_1, \ldots, g_r$ as $G = < g_1, \ldots, g_r >$ and we shall denote the number of generators of G by $|G|$. A group G is a **torsion group** if every element of G is of finite order. G is **torsion free** if no element other than the identity is of finite order. In an abelian group the set T of all finite order elements form a subgroup called the **torsion subgroup of G**. Any lattice under vector addition, L, is torsion free, *i.e.* $L \cong \mathbb{Z}^n$ for some n, and can be written as $L = \langle\alpha_i\rangle$, where the α_i form a basis of L.

In general if $L' \subset L$ is a sublattice of L then L' is a normal subgroup of L because L is abelian. We can therefore form the quotient group $Q \equiv \frac{L}{L'}$. Q will also be a finitely generated abelian group but in general it will not be torsion free. Of necessity (Fundamental Theorem of Finitely Generated Abelian Groups) it is of the form

$$\frac{L}{L'} \cong F \times T,$$

where

1. $F \cong \mathbb{Z}^m$ is a finitely generated torsion free abelian group, and m is known as the **Betti number of F**.
2. $T \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \ldots \times \mathbb{Z}_{(p_n)^{r_n}}$ where the p_i are (not necessarily distinct) prime numbers, or
   $T \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r}$ where m_i divides m_{i+1} and the number of m_is is known as the **torsion coefficient of T**.

We write

$$\frac{L}{L'} = \langle\beta_1, \ldots, \beta_m; \tau_1, \ldots, \tau_n\rangle$$

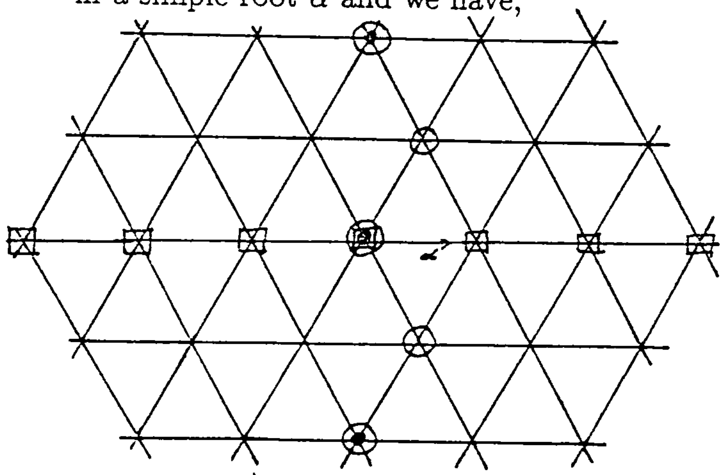
where the $\{\tau_i\}$ generate the torsion subgroup. Let $\Gamma \subset L$ be the lattice generated by the $\{\beta_i\}$. This means that we have the following coset decomposition of L with respect to L′,

$$\mathrm{L} = \bigcup_{\beta\in\Gamma} \bigcup_{i=1}^{n} (\beta + \tau_i + \mathrm{L}').$$

The coset representatives $\{\beta + \tau_i \mid i = 1,...,n,\ \beta \in \Gamma\}$ form a maximal set of vectors in L such that they do not differ from each other by an element of L′. By convention we take $\tau_1 = 0$.

Firstly we single out the sublattice $\mathrm{N} = (1 - \mathrm{P}_0)\mathrm{V} \cap \Lambda_R \cong \mathbb{Z}^{\mathrm{rank}\,g - |\mathrm{P}_0\Lambda_R|}$ where V is the real span of the roots of Λ_R. Let $\Lambda_R^0 \equiv \mathrm{P}_0\mathrm{V} \cap \Lambda_R$ be the sublattice of the root lattice left fixed by σ, and define $\widehat{\Lambda_R} \equiv \frac{\Lambda_R}{\Lambda_R^0}$. In general an element of $\frac{\Lambda_R}{\mathrm{N}}$ can be written as $\alpha^0 + \gamma_j + \mathrm{N}$ where $\alpha^0 \in \Lambda_R^0$ and $\gamma_j \in \widehat{\frac{\Lambda_R}{\mathrm{N}}}$. Let $\mathrm{L} = |\ \widehat{\frac{\Lambda_R}{\mathrm{N}}}\ |$.

For example for the second order automorphism of A_2 consisting of a Weyl reflection in a simple root α and we have,



• denotes an element of Λ_R^0,

○ denotes an element of $\frac{\Lambda_R}{\mathrm{N}}$,

□ denotes an element of N.

In fact $\frac{\Lambda_R}{\mathrm{N}} \cong \mathrm{P}_0\Lambda_R \cong \mathbb{Z}^{|\mathrm{P}_0\Lambda_R|}$ where the first isomorphism is given by $\alpha^0 + \gamma_j \mapsto \alpha^0 + \mathrm{P}_0(\gamma_j)$. (In the above example $\frac{\Lambda_R}{\mathrm{N}} \cong \mathbb{Z}$). Therefore

$$\frac{\Lambda_R}{\mathrm{N}} \cong \langle \alpha_{0\ i} \rangle\,,$$

where $\alpha_{0\ i}$ is a basis for $\mathrm{P}_0\Lambda_R$. With a slight abuse of notation we can write

$$\frac{\Lambda_R}{\mathrm{N}} \cong \langle \mathrm{P}_0\alpha_i \rangle\,,$$

where α_i are simple roots. In general some of the $\mathrm{P}_0\alpha_i$ will be equal to zero and others will be redundant.

We will use this isomorphism later to identify the cosets of $\frac{\Lambda_R}{N}$ with the momentum eigenstates $| \alpha_0\rangle$, $\alpha_0 \in P_0\Lambda_R$, via

$$\alpha^0 + \gamma_j \mapsto | \alpha^0 + P_0(\gamma_j) > .$$

Notice that we will have to introduce a degenerate vacuum if the commutator map $C(\alpha,\beta)$ is non–trivial on N, by which we mean that we do not have $C(\alpha,\beta) = 1 \ \forall \alpha, \beta \in$ N. This is because for all α in N, $\alpha_0 \equiv P_0\alpha = 0$ so that $\hat{C}(\alpha) = C_\alpha$ and $\alpha.\beta_0 = 0$ $\forall \beta_0 \in P_0\Lambda_R$. Therefore we cannot use the space of momentum eigenstates to form a representation of the cocycle operators on N.

Next we consider the sublattice $M \equiv (1-\sigma)\Lambda_R \subset N$. M is also isomorphic to $\mathbb{Z}^{\mathrm{rank}\, g - |P_0\Lambda_R|}$ and $\frac{N}{M}$ is a finite group [5]. Thus if X is any sublattice of N such that $M \subset X \subset N$ then $\frac{N}{X}$ will also be a finite group. In particular we choose a maximal sublattice $A \subset N$ such that $C(\alpha,\beta) = 1 \ \forall \alpha,\beta \in A$. Such a lattice is not uniquely defined but all the choices are isomorphic and give equivalent constructions of the cocycle operators. Obviously $M \subset A \subset N$ so that both $\frac{A}{M}$ and $\frac{N}{A}$ are torsion groups. We let

$$\begin{aligned} \frac{A}{M} &= < a_1, \ldots\ldots, a_{A_M} > \qquad & A_M &= \left|\frac{A}{M}\right|, \\ \frac{N}{A} &= < n_1, \ldots\ldots, n_{N_A} > \qquad & N_A &= \left|\frac{N}{A}\right|. \end{aligned}$$

We are now in a position to construct the cocycle matrices C_α.

A is a maximal sublattice on which the cocycle matrices commute,

$$C_\alpha C_\beta = C_\beta C_\alpha \qquad \forall \alpha,\beta \in A.$$

Thus we only need to choose the C_α to be a one–dimensional abelian representation on A. However we are restricted in our choice of representation by (6.16), which together with (6.17) gives

$$\begin{aligned} C_{(1-\sigma)\alpha} &= \frac{C_\alpha C_{-\sigma(\alpha)}}{\varepsilon_c(\alpha,-\sigma(\alpha))}, \\ &= \frac{C_\alpha C_{-\alpha}}{\varepsilon_c(\alpha,-\sigma(\alpha))}\frac{\omega^{\tilde{\alpha}.p-\frac{\tilde{\alpha}.\alpha}{2}}}{\psi_{-\alpha}}, \\ &= C_0\frac{\varepsilon_c(\alpha,-\alpha)}{\varepsilon_c(\alpha,-\sigma(\alpha))}\frac{\omega^{\tilde{\alpha}.p-\frac{\tilde{\alpha}.\alpha}{2}}}{\psi_{-\alpha}}, \\ &= \psi_\alpha C_0 \varepsilon_c(\alpha,-(1-\sigma)\alpha)\omega^{\tilde{\alpha}.p-\frac{\tilde{\alpha}.\alpha}{2}}. \end{aligned}$$

As $C_0 = \mathbb{1}$ we must choose the cocycle operator on M to be given by

$$C_{(1-\sigma)\alpha} = \psi_\alpha \varepsilon_c(\alpha, -(1-\sigma)\alpha)\omega^{\tilde{\alpha}.p - \frac{\tilde{\alpha}.\alpha}{2}}, \tag{6.21}$$
$$= \psi_\alpha \varepsilon_c(\alpha, -(1-\sigma)\alpha)\omega^{-\frac{\tilde{\alpha}.\alpha}{2}} \sum_{\beta_0 \in P_0\Lambda_R} |\,\beta_0\rangle \omega^{\tilde{\alpha}.\beta_0} \langle \beta_0\,| \,.$$

The momentum eigenstates are assumed to be orthonormal.

Lemma (6.1) :

1. $\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\sigma(\alpha),\sigma(\beta))} = \frac{\varepsilon_c(\alpha,\beta)}{\varepsilon_c(\sigma(\alpha),\sigma(\beta))} = \pm 1$ so that $\psi_\alpha\psi_\beta = \frac{\varepsilon_c(\sigma(\alpha),\sigma(\beta))}{\varepsilon_c(\alpha,\beta)}\psi_{\alpha+\beta}$.
2. $C((1-\sigma)\alpha, \beta) = \omega^{\tilde{\alpha}.\beta}$.

Proof :

1. The first equality follows from the fact that,

   (a) $\varepsilon(\alpha,\beta) = \varepsilon'(\alpha,\beta)\varepsilon_c(\alpha,\beta)$,

   (b) $\varepsilon'(\sigma(\alpha),\sigma(\beta)) = \varepsilon'(\alpha,\beta)$,

   where $\varepsilon'(\alpha,\beta) \equiv \prod_{r=1}^{N-1}(1-\omega^{-r})^{\sigma^r(\alpha).\beta}$. In addition $\varepsilon(\alpha,\beta) = \pm 1$ for simply laced algebras so that

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\sigma(\alpha),\sigma(\beta))} = \frac{\varepsilon(\sigma(\alpha),\sigma(\beta))}{\varepsilon(\alpha,\beta)} = \pm 1.$$

2.

$$C(\sigma(\alpha),\beta) = (-1)^{\tilde{\alpha}.\beta}\omega^{\sum_{r=0}^{N-1} r\sigma^{r+1}(\alpha).\beta},$$
$$= (-1)^{\tilde{\alpha}.\beta}\omega^{\sum_{r=0}^{N-1}(r-1)\sigma^{r}(\alpha).\beta)},$$
$$= \omega^{\tilde{\alpha}.\beta}C(\alpha,\beta)\,.$$

It is a relatively simple but uninspiring exercise to prove that (6.21) does indeed satisfy (6.16) and (6.17) using the results of this Lemma. Further for an arbitrary $\beta \in \Lambda_R$ we have

$$\begin{aligned}
\psi_\beta \hat{C}(\sigma(\beta)) &= \psi_\beta \hat{C}(\beta - (1-\sigma)\beta)\,, \\
&= \psi_\beta \frac{\hat{C}(\beta)\hat{C}(-(1-\sigma)\beta)}{\varepsilon_c(\beta, -(1-\sigma)\beta)}\,, \\
&= \psi_\beta \hat{C}(\beta)\psi_{-\beta} C_0 \frac{\varepsilon_c(-\beta, (1-\sigma)\beta)}{\varepsilon_c(\beta, -(1-\sigma)\beta)} \omega^{-\frac{\bar{\beta}.\beta}{2} - \beta.p}\,, \\
&= \hat{C}(\beta)\omega^{-\frac{\bar{\beta}.\beta}{2} - \beta.p}\,.
\end{aligned}$$

So with this choice of $\hat{C}$ on M we have fixed (6.17) for the whole lattice as long as we make sure the extension to the whole lattice satisfies (6.16).

To extend the definition of $\hat{C}$ to A we need an arbitrary homomorphism $\chi : \frac{A}{M} \to \mathbb{C} - \{0\}$. As $\frac{A}{M}$ is a finite abelian group there are A_M such inequivalent homomorphisms. Given such a homomorphism we can take

$$\hat{C}(a_i + (1-\sigma)\alpha) = \chi(a_i)\hat{C}((1-\sigma)\alpha)\,.$$

For simplicity we take χ to be the trivial homomorphism, $\chi(a_i) = 1 \ \forall i$. Obviously (6.16) and (6.17) still hold.

Now when we come to extend the definition of $\hat{C}$ to Λ_R we cannot repeat the procedure as the commutator map $C(\alpha, \beta)$ is not trivial on Λ_R. We must therefore use the representation of A to form an induced representation of Λ_R. Before proceeding we shall briefly explain what is meant by an induced representation. Let G be a group and H a subgroup of G. Further let

$$G = x_1 H \cup x_2 H \cup \ldots\ldots\ldots\ldots$$

be coset decomposition of G with respect to H, where $\{x_i\}$ is a set of coset representatives. Then given an n–dimensional representation $\psi : H \to GL_n(\mathbb{C})$ of H we can form a representation of G as follows.

Define ϕ on all of G by

$$\phi(x) = \begin{cases} \psi(x) & x \in H, \\ 0 & x \notin H \end{cases} \qquad 0 = n \times n \text{ zero matrix.}$$

The **induced representation** of G is then given by the set of block matrices

$$\Phi(\mathrm{g}) = (\Phi_{ij}(\mathrm{g})) \equiv (\phi(x_i^{-1}\mathrm{g}x_j)).$$

<u>Note :</u> The matrix $\Phi(\mathrm{g})$ has only one non-zero block in each row and column of blocks.

In our case we have a one-dimensional representation of A with which we want to induce a representation of Λ_R. The coset decomposition of Λ_R with respect to A is in general infinite,

$$\Lambda_R = \bigcup_{\alpha^0 \in \Lambda_R^0} \bigcup_{j=1}^{\mathrm{L}} \bigcup_{k=1}^{\mathrm{N_A}} (\alpha^0 + \gamma_j + \mathrm{n}_k + \mathrm{A}),$$

where as before $\{\alpha^0 + \gamma_j\}$ are coset representatives of a decomposition of Λ_R with respect to N whilst the $\{\mathrm{n}_k\}$ are coset representatives of a decomposition of N with respect to A. As was stated earlier $\frac{\Lambda_R}{\mathrm{N}} \cong \mathrm{P}_0\Lambda_R$ so that we can use the momentum eigenstates as a representation space of this part of $\hat{\mathrm{C}}$. However $\mathrm{n}_k \in \mathrm{N}$ so that we must introduce an $\mathrm{N_A}$-dimensional space to represent this part of $\hat{\mathrm{C}}$ on. We will let $\{\mid k\rangle \mid k = 1, ..., \mathrm{N_A}\}$ be a basis of this space where the state $\mid k\rangle$ corresponds to the coset representative n_k. We assume that this basis is orthonormal, $< i \mid j > = \delta_{ij}$.

Firstly noticing that,

$$\hat{\mathrm{C}}(\alpha)^{-1} = \frac{\hat{\mathrm{C}}(-\alpha)}{\varepsilon_c(\alpha, -\alpha)}.$$

and then setting $\beta = \alpha - \alpha^0 + \beta^0 - \gamma_j + \gamma_l - \mathrm{n}_k + \mathrm{n}_m$, we have;

$$\hat{\mathrm{C}}(\alpha) = \begin{cases} \displaystyle\sum_{\alpha^0, \beta^0 \in \Lambda_R^0} \sum_{j,l=1}^{\mathrm{L}} \sum_{k,m=1}^{\mathrm{N_A}} \frac{\varepsilon_c(-(\alpha^0 + \gamma_j + \mathrm{n}_k), \alpha + \beta^0 + \gamma_l + \mathrm{n}_m)\varepsilon_c(\alpha, \beta^0 + \gamma_l + \mathrm{n}_m)}{\varepsilon_c(\alpha^0 + \gamma_j + \mathrm{n}_k, -(\alpha^0 + \gamma_j + \mathrm{n}_k))} & \\ \times\ \mathrm{C}_\beta \mid k\rangle \mid \alpha^0 + \mathrm{P}_0(\gamma_j)\rangle\langle\beta^0 + \mathrm{P}_0(\gamma_l) \mid \langle m \mid & \beta \in \mathrm{A}, \\ 0 & \beta \notin \mathrm{A}. \end{cases}$$

Therefore

$$\hat{\mathrm{C}}(\alpha) = \sum_{\beta \in \mathrm{A}} \sum_{\beta^0 \in \Lambda_R^0} \sum_{l=1}^{\mathrm{L}} \sum_{m=1}^{\mathrm{N_A}} \varepsilon_c(\beta - \alpha - (\beta^0 + \gamma_l + \mathrm{n}_m), \beta)\varepsilon_c(\alpha, \beta^0 + \gamma_l + \mathrm{n}_m)\mathrm{C}_\beta$$
$$\times \mid k\rangle \mid \mathrm{P}_0(\alpha) + \beta^0 + \mathrm{P}_0(\gamma_l)\rangle\langle\beta^0 + \mathrm{P}_0(\gamma_l) \mid \langle m \mid .$$

Let $\mu = \beta^0 + \gamma_l$ and $\alpha = \eta + \mathrm{n}_a + a$ where $\eta \in \frac{\Lambda_R}{\mathrm{N}}$, $\mathrm{n}_a \in \frac{\mathrm{N}}{\mathrm{A}}$ and $a \in \mathrm{A}$. Then

$\beta \in \mathrm{A}$ implies that $\mathrm{n}_a - \mathrm{n}_k + \mathrm{n}_m = 0$ so that the final form of the cocycle operator is

$$\hat{\mathrm{C}}(\alpha) = \sum_{\mu \in \frac{\Lambda_R}{\mathrm{N}}} \sum_{m=1}^{\mathrm{N_A}} \sum_{\beta \in \mathrm{A}} \varepsilon_c(\beta-\alpha-\mu-\mathrm{n}_m,\beta)\varepsilon_c(\alpha,\mu+\mathrm{n}_m)\mathrm{C}_\beta \mid m+a\rangle \mid \mathrm{P}_0(\alpha+\mu)\rangle\langle \mathrm{P}_0(\mu) \mid \langle m \mid .$$

where $\mid m+a\rangle$ denotes the state corresponding to the coset $\mathrm{n}_m + \mathrm{n}_a$.

In general we cannot simply disentangle the momentum and vacuum degeneracy pieces of the cocycle operators. Generically we have $\mathrm{M} \subseteq \mathrm{A} \subseteq \mathrm{N} \subseteq \Lambda_R$ and $\Lambda_R^0 \subseteq \mathrm{P}_0\Lambda_R \subseteq \Lambda_R$ but we shall now look at some special cases.

1. $\alpha \in \mathrm{A}$:

$$\hat{\mathrm{C}}(\alpha) = \sum_{\mu \in \frac{\Lambda_R}{\mathrm{N}}} \sum_{m=1}^{\mathrm{N_A}} \sum_{\beta \in \mathrm{A}} \varepsilon_c(\beta-\alpha-\mu-\mathrm{n}_m,\beta)\varepsilon_c(\alpha,\mu+\mathrm{n}_m)\mathrm{C}_\beta \mid m\rangle \mid \mathrm{P}_0(\mu)\rangle\langle \mathrm{P}_0(\mu) \mid \langle m \mid .$$

Therefore the matrix is diagonal, as expected.

2. If σ only fixes the origin then $\Lambda_R^0 = \mathrm{P}_0\Lambda_R = \{0\}$ and $\mathrm{N} = \Lambda_R$, thus we have

$$\hat{\mathrm{C}}(\alpha) = \sum_{m=1}^{\mathrm{N_A}} \sum_{\beta \in \mathrm{A}} \varepsilon_c(\beta-\alpha-\mathrm{n}_m,\beta)\varepsilon_c(\alpha,\mathrm{n}_m)\mathrm{C}_\beta \mid m+a\rangle \mid \langle m \mid . \tag{6.22}$$

and $\mathrm{C}_{(1-\sigma)\eta} = \psi_\eta \varepsilon_c(\eta, -(1-\sigma)\eta)$. But in this case $(1-\sigma)$ is invertible so that

$$\begin{aligned}\hat{\mathrm{C}}(\alpha) &= \sum_{m=1}^{\mathrm{N_A}} \sum_{\beta \in \mathrm{A}} \varepsilon_c(\beta-\alpha-\mathrm{n}_m,\beta)\varepsilon_c(\alpha,\mathrm{n}_m)\varepsilon_c\left((1-\sigma)^{-1}\beta,-\beta\right)\psi_{(1-\sigma)^{-1}\beta} \mid m+a\rangle\langle m \mid \\ &= \mathrm{C}_\alpha .\end{aligned}$$

$\hat{\mathrm{C}}(\alpha)$ is a pure matrix and does not act on a momentum eigenspace. For NFPAs this simplification occurs for all sectors of the string theory.

3. $\sigma = \mathbb{1}$, $\Lambda_R^0 = \Lambda_R$, $\mathrm{A} = \mathrm{N} = \{0\}$ and $\varepsilon_c = \varepsilon$ **(untwisted case)**:

$$\begin{aligned}\hat{\mathrm{C}}(\alpha) &= \sum_{\mu \in \Lambda_R} \varepsilon(-\alpha-\mu,0)\varepsilon(\alpha,\mu)\mathrm{C}_0 \mid \alpha+\mu\rangle\langle\mu \mid, , \\ &= \sum_{\mu \in \Lambda_R} \varepsilon(\alpha,\mu) \mid \alpha+\mu\rangle\langle\mu \mid , \\ &= e^{i\alpha.q} \sum_{\mu \in \Lambda_R} \varepsilon(\alpha,\mu) \mid \mu\rangle\langle\mu \mid .\end{aligned}$$

We therefore have a generalisation of a previous construction given in [46].

4. A = N :

$$\hat{C}(\alpha) = \sum_{\mu \in \frac{\Lambda_R}{N}} \sum_{\beta \in A} \varepsilon_c(\beta - \alpha - \mu, \beta)\varepsilon_c(\alpha, \mu) C_\beta \mid P_0(\alpha + \mu)\rangle\langle P_0(\mu) \mid .$$

In this case there is no vacuum degeneracy and the cocycle operators can be represented purely in terms of operators acting on the momentum eigenspace.

6.4 Some examples of twisted vertex operators and cocycles.

If we take σ to be the identity automorphism, $\sigma = \mathbb{1}$, then $Q_1(z) = Q(z)$, $\alpha_0 = \alpha \ \forall \alpha \in \Lambda_R$ and N= 1 so that

$$\begin{aligned} V_1(\alpha, z) &= : e^{i\alpha . Q(z)} : C_\alpha , \\ &= V(\alpha, z) . \end{aligned}$$

In addition $C(\alpha,\beta) = (-1)^{\alpha.\beta}$ so that the general twisted vertex operator construction subsumes the Frenkel–Kac–Segal construction.

Another special case occurs when we take σ to be an automorphism which only fixes the origin. In this case $\alpha_0 = 0 \ \ \forall \alpha \in \Lambda_R$ and $C(\alpha,\beta) = \omega^{\sum_{r=1}^{N-1} r\sigma^r(\alpha).\beta}$,

$$V_\sigma(\alpha, z) = N^{-\frac{\alpha^2}{2}} z^{-\frac{\alpha^2}{2}} : e^{i\alpha . Q_\sigma(z)} : C_\alpha ,$$

and

$$Q_\sigma(z) = i \sum_{r \in \frac{1}{N}Z} \frac{b_r}{r} z^{-r} .$$

As was explained in the last section the cocycles are pure matrices which act solely on the space due to vacuum degeneracy, see (6.22). If in addition the automorphism is a NFPA then all the sectors of the corresponding string theory have this simplified form.

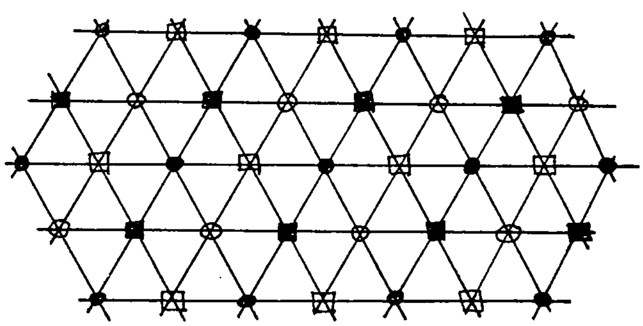
The only NFPA which is shared by all the algebras is the inversion in the origin. It is in fact the only NFPA for odd rank algebras. This gives $C(\alpha,\beta) = (-1)^{\alpha.\beta}$ and

$$Q_{-1}(z) = i \sum_{r \in \frac{1}{2}Z} \frac{b_r}{r} z^{-r} .$$

It retrieves the construction that first appeared in [8] without the cocycle operators. The cocycle matrices for it can be constructed from the ordinary γ matrices [47]. For example in A_2 we find that $N = c_\sigma = 2$ and the cocyle matrices are just the Pauli spin matrices associated with the lattice in the following way [49]

1. $c_\gamma = \mathbb{1} \quad \gamma \in 2\Lambda_R$,
2. $c_\gamma = \sigma_1 \quad \gamma \in 2\Lambda_R + \alpha$,
3. $c_\gamma = \sigma_2 \quad \gamma \in 2\Lambda_R + \beta$,
4. $c_\gamma = \sigma_3 \quad \gamma \in 2\Lambda_R + \alpha + \beta$,

where α, β are the simple roots of A_2 *i.e.*



● $= \mathbb{1}$,

□ $= \sigma_1$,

○ $= \sigma_2$,

■ $= \sigma_3$.

In general as $\sigma(\alpha) = -\alpha$ and $\Sigma(E_\alpha) = \psi_\alpha E_{-\alpha}$ there will be one invariant element for each positive root and none in the Cartan subalgebra, so

$$\begin{aligned} \dim g_0 &= |\Phi_g^+| , \\ &= \frac{1}{2}(\dim g - \operatorname{rank} g) . \end{aligned}$$

(See also section 7.2). It follows that $\dim g_1 = |\Phi_g^+| + \operatorname{rank} g$. The resulting invariant subalgebras are well known. We list them here along with the vacuum degeneracy of the corresponding twisted vertex operator representation [6,49];

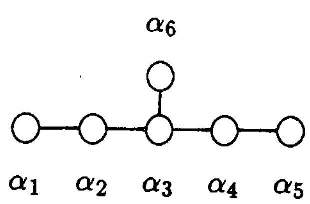
Table 6.1 : *Invariant subalgebras and vacuum degeneracies for second order NFPAs.*

g	g_0	Vacuum degeneracy
su(n)	so(n)	spinor $2^{\frac{n}{2}}$ $\quad n \in 2\mathbb{Z}$ spinor $2^{\frac{(n-1)}{2}}$ $\quad n \in 2\mathbb{Z}+1$
so(2n)	so(n)$\oplus$so(n)	(spinor, singlet)
E_6	sp(4)	$\underline{8}$
E_7	su(8)	$\underline{8}$
E_8	so(16)	$\underline{16}$

The number of orbifold fixed points in each case is given by $F = \det(1-\sigma) = 2^{\mathrm{rank}\, g}$. Notice that $\mathrm{rank}\, g_0 < \mathrm{rank}\, g$ iff σ is outer, which by Lemma (4.3) is true for A_n, D_n $n \in 2\mathbb{Z}+1$ and E_6.

Let us now look at a couple of third order NFPAs.

Example 1 : E_6, Conjugacy class no. 21. With respect to a basis of simple roots an example of σ is given by,



$$\sigma = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 0 & 3 \\ 0 & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$F = \det(1-\sigma) = 27$, therefore $c_\sigma^2 \leq 27$ and as $R_M \equiv |\frac{R}{M}| \in \mathbb{Z}$ we must have one of the following

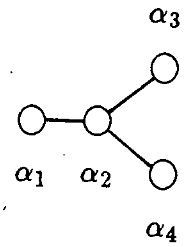
1. $R_M = 3$ and $c_\sigma = 3$,

2. $R_M = 27$ and $c_\sigma = 1$.

In fact as $N = \Lambda_R$ and $A \neq \Lambda_R$ we must have $c_\sigma = 3$. We can choose $A = \{m_i\alpha_i \,|\, m_2,\ m_4,\ m_5,\ m_6 \in \mathbb{Z},\ m_1,\ m_3 \in 3\mathbb{Z}\}$ and $\frac{N}{A} = \{[n\alpha_1 + m\alpha_3] \mid n,\ m = 0,\ 1,\ 2\} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

In this case $g_0 \cong A_8$ [su(9)] and the vacuum forms a $\underline{9}$.

Example 2 : D_4, Conjugacy class no. 21. With respect to a basis of simple roots an example of σ is given by,



$$\sigma = \begin{pmatrix} -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

F $= \det(1-\sigma) = 9$, thus either

1. $R_M = 1$ and $c_\sigma = 3$, or
2. $R_M = 9$ and $c_\sigma = 1$.

Again N $= \Lambda_R$ and A $\neq \Lambda_R$ so that $c_\sigma = 3$ and $\frac{N}{A} \cong \mathbb{Z}_3$. $g_0 \cong A_2$ [su(3)] and the vacuum is a $\underline{3}$.

6.5 Comments on the vaccum degeneracy and orbifold fixed points.

Firstly let us establish the equivalence between the notations of [16,44] and [5]. In the former papers, whose notation is more relevant to the orbifold construction, the sets

$$M_\sigma \equiv \{x \in V \mid (1-\sigma)x \in \Lambda_R\}$$

and

$$M'_\sigma \equiv \{x \in M_\sigma \mid \Psi(x,y) = 1 \;\forall y \in M_\sigma\}$$

are considered where

$$\Psi(x,y) \equiv e^{2\pi i x.(1-\sigma)y}.$$

Lemma (6.2) : We have the following isomorphisms

$$\frac{M_\sigma}{P_0 V} \cong N, \; \frac{M'_\sigma}{P_0 V} \cong R.$$

Proof : Obviously $(1-\sigma) : M_\sigma \to N$ is a homomorphism as it is a linear map. $\mathrm{Im}(1-\sigma) = N$ because if we take σ_* to be the restriction of σ to N, that is $\sigma_* =$

$(1-\mathrm{P}_0)\sigma$, then $\det(1-\sigma_*) \neq 0$ so that $(1-\sigma_*)$ is invertible on N. We can therefore choose $x = (1-\sigma_*)^{-1}\alpha$ as an element of M_σ that maps onto α. The kernel of $(1-\sigma)$ is $\mathrm{P}_0\mathrm{V}$, so by the First Isomorphism Theorem we have the first result. The second follows similarly by looking at $(1-\sigma)$ acting on M'_σ.

In particular we therefore have

$$\frac{\mathrm{M}_\sigma}{\mathrm{M}'_\sigma} \cong \frac{\mathrm{N}}{\mathrm{R}}\,.$$

In addition if we let $\alpha = (1-\sigma)x + \alpha_0$, $\beta = (1-\sigma)y$ we have $x.(1-\sigma)y = x.\beta$. Let

$$\mathrm{X} = -\frac{1}{\mathrm{N}}\sum_{r=0}^{\mathrm{N}-1} r\sigma^r.$$

Then

$$\begin{aligned}
\mathrm{X}.(1-\sigma) &= -\frac{1}{\mathrm{N}}\{\sum_{r=0}^{\mathrm{N}-1} r\sigma^r - r\sigma^{r+1}\},\\
&= -\frac{1}{\mathrm{N}}\{\sum_{r=0}^{\mathrm{N}-1} \sigma^r - (\mathrm{N}-1)\},\\
&= 1 \qquad\qquad \text{on N},
\end{aligned}$$

as $\sum_{r=0}^{\mathrm{N}-1} \sigma^r = 1$ on N. Thus $\mathrm{X} = (1-\sigma_*)^{-1}$. Therefore $x = \mathrm{X}(\alpha) + \alpha_0$ and

$$\Psi(x,y) = \mathrm{C}(\alpha,\beta).$$

Vacuum degeneracy :

It is easy to show that C is an *alternating bilinear form* on N, that is

$$\mathrm{C}(\alpha,\beta) = \frac{1}{\mathrm{C}(\beta,\alpha)} \qquad \forall\, \alpha,\ \beta \in \mathrm{N},$$

or alternatively $\mathrm{C}(\alpha,\alpha) = 1\ \forall\ \alpha \in \mathrm{N}$. Let $\mathrm{R} \subset \mathrm{N}$ be the **radical** of C on N,

$$\mathrm{R} \equiv \{\alpha \in \mathrm{N} \mid \mathrm{C}(\alpha,\mathrm{N}) = 1\}\,.$$

Now as $\mathrm{C}(\alpha,\beta)$ is alternating on N, $\mathrm{N_R} \equiv |\frac{\mathrm{N}}{\mathrm{R}}|$ is the square of a positive integer c_σ called the **defect** of σ [16]. In the notation of [5] this corresponds to the fact that

$N_R = N_A A_R$ and $N_A = A_R$, which explains our previous identification of N_A $(= c_\sigma)$ with the **vacuum degeneracy** .

Let us look at the lattices involved in a little more detail. Firstly

$$N = (1 - P_0)\Lambda_R \cap \Lambda_R \,.$$

Secondly if $\alpha \in R$ then we have $-\frac{1}{N}\sum_{r=0}^{N-1} r\sigma^r(\alpha).\beta \in \mathbb{Z} \;\forall\; \beta \in N$, therefore

$$-\frac{1}{N}\sum_{r=0}^{N-1} r\sigma^r(\alpha) \in N^* \qquad \text{and,}$$
$$\alpha \in (1-\sigma)N^* \cap N \,.$$

But $N^* = \{\alpha + v_0 \mid \alpha \in \Lambda_W,\ v_0 \in P_0 V\}$ and therefore

$$R = (1-\sigma)\Lambda_W \cap \Lambda_R.$$

It is not too difficult to verify that we do indeed have $M \subset R \subset N$ namely,

$$(1-\sigma)\Lambda_R \subset (1-\sigma)\Lambda_W \cap \Lambda_R \subset (1-P_0)\Lambda_R \cap \Lambda_R \,.$$

The vacuum degeneracy is given by

$$c_\sigma = \left|\frac{N}{R}\right|^{\frac{1}{2}} = \left|\frac{(1-P_0)\Lambda_R \cap \Lambda_R}{(1-\sigma)\Lambda_W \cap \Lambda_R}\right|^{\frac{1}{2}} .$$

For the special case of a σ that leaves only the origin fixed we have $N = \Lambda_R$ and

$$c_\sigma = \left|\frac{\Lambda_R}{(1-\sigma)\Lambda_W \cap \Lambda_R}\right|^{\frac{1}{2}} .$$

In [16] it is shown that

$$\begin{aligned} c_\sigma &= [\det(1-\sigma_*)]^{\frac{1}{2}} \frac{\mathrm{Vol} P_0 \Lambda_R}{\mathrm{Vol}\Lambda_R} \,, \\ &= \left|\mathrm{torsion}\frac{\Lambda_R}{(1-\sigma)\Lambda_W}\right|^{\frac{1}{2}} , \end{aligned}$$

where Vol stands for volume.

[16] also gives all the possible values of c_σ for representations formed by using inner automorphisms of simply laced algebras,

$$\begin{aligned}
&A_n: \ 1, \\
&D_n: \ 2^r \text{ where } r \leq \left[\frac{n-2}{2}\right] \ (\text{for } D_4 \text{ this means } c_\sigma = 1, 2), \\
&E_6: \ 1, 2, \text{ or } 3, \\
&E_7: \ 1, 2, 3, 4 \text{ or } 8, \\
&E_8: \ 1, 2, 3, 4, 5, 6, 8, 9, \text{ or } 16.
\end{aligned}$$

Fixed subspaces of the orbifold :

If $v \in V$ corresponds to a fixed point singularity of the orbifold then $v = \sigma v + L$ where $L \in \Lambda_R$ and therefore $(1-\sigma)v \in \Lambda_R$ *i.e.* $v \in M_\sigma$. But $(1-P_0)[(1-\sigma)v] = (1-\sigma)v$ so in fact we must have

$$(1-\sigma)v \in (1-P_0)\Lambda_R \cap \Lambda_R = N$$

and thus

$$v \in (1-\sigma_*)^{-1}N + P_0V.$$

In general the singularities are $\dim P_0V$–dimensional and there are

$$F = \left|\frac{(1-\sigma_*)^{-1}N}{N}\right| \quad \text{of them.}$$

However

$$\frac{(1-\sigma_*)^{-1}N}{N} \cong \frac{N}{(1-\sigma_*)N}$$

and

$$(1-\sigma_*)N = (1-\sigma)N.$$

So

$$
\begin{aligned}
F &= \left| \frac{N}{(1-\sigma)N} \right| , \\
&= \det(1-\sigma_*) , \\
&= \prod_{\substack{i=1 \\ n_i \neq 0}}^{\operatorname{rank} g} (1-\omega^{n_i}) , \\
&= \prod_{\substack{i=1 \\ \theta_i \neq 0}}^{\operatorname{rank} g} 2(1-\cos\theta_i) .
\end{aligned}
$$

Again in the case of a σ which only fixes the origin we have

$$
\begin{aligned}
F &= \left| \frac{\Lambda_R}{(1-\sigma)\Lambda_R} \right| , \\
&= 2^{\operatorname{rank} g} \prod_{i=1}^{\operatorname{rank} g} (1-\cos\theta_i) ,
\end{aligned}
$$

0–dimensional, or point, singularities in the orbifold. In this case $F = |\det(1-\sigma)|$.

We can now relate the number of fixed 'points' to the vacuum degeneracy. Firstly notice that $(1-\sigma)N \subset M \subset R$ so that it makes sense to consider

$$
\frac{R}{(1-\sigma)N} .
$$

The number of fixed 'points' is given by

$$
\begin{aligned}
F &= \left| \frac{N}{(1-\sigma)N} \right| , \\
&= \left| \frac{N}{A} \right| \left| \frac{A}{R} \right| \left| \frac{R}{(1-\sigma)N} \right| ,
\end{aligned}
$$

and we know that $N_A = A_R = c_\sigma$ so,

$$
F = c_\sigma^2 \left| \frac{R}{(1-\sigma)N} \right| .
$$

Hence in general $F \geq c_\sigma^2$. In the special case of no fixed points but the origin we have

$$
F = c_\sigma^2 \left| \frac{R}{M} \right| .
$$

Let us concentrate on this case. Generically $M \subseteq R \subseteq N \subseteq \Lambda_R$ but there are some special cases.

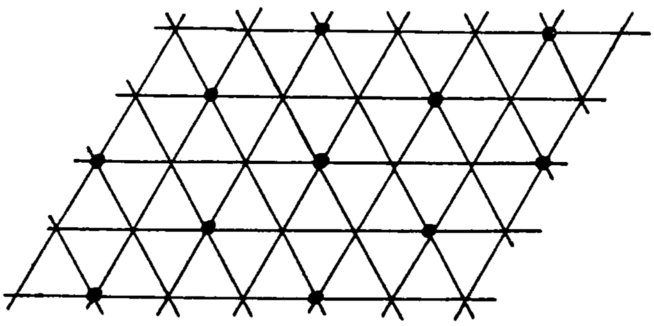
1. $M = R = (1-\sigma)\Lambda_R$ if $\Lambda_W = \Lambda_R$, *i.e.* if the lattice is **self dual** (for eg E_8),

$$F = c_\sigma^2$$

therefore

$$c_\sigma = \sqrt{|\det(1-\sigma)|}\,.$$

2. $R = N = \Lambda_R$ if $(1-\sigma)\Lambda_W = \Lambda_R$. That is σ is primitive (*i.e.* $\det(1-\sigma) = \det A$), see p 103. In this case $c_\sigma = 1$ and $F = |Z_g|$ where Z_g is the centre of g. This occurs for the third order NFPA of A_2, the rotation through $\frac{2\pi}{3}$, as $\sigma\Lambda_3 = \Lambda_3$, $\sigma\Lambda_{\bar{3}} = \Lambda_{\bar{3}}$, $\sigma\Lambda_R = \Lambda_R$ and $(1-\sigma)v \in \Lambda_R$ if $v \in \Lambda_3$ or $\Lambda_{\bar{3}}$.



$N = \Lambda_R$, $M = \{\bullet\}$.

$\frac{N}{M} \cong \mathbb{Z}_3$ and $F = 3$.

7. Invariant Subalgebras and other Results.

We start this chapter by examining an alternative construction of the graded representation of a Kac–Moody algebra corresponding to an inner automorphism. This is known as a **shifted vertex operator representation** as it corresponds to the shifted picture of the corresponding Lie algebra automorphism. We then go on to show how by comparing the algebra automorphism in the shifted and twisted pictures and looking at the corresponding Kac–Moody algebra representations we are able to determine the invariant subalgebras and vacuum degeneracies. We finish by giving the results of our calculations for the simply laced exceptional Lie algebras . The results for E_8 were previously given in [7]. Similar calculations for E_8 were also considered in [44].

7.1 Shifted vertex operators.

There is an alternative way to generalise the 'homogeneously' graded construction given in Section 6.1. This is obtained by altering the zero–mode space by considering the momentum to lie in a different coset of the root lattice in V, namely $\Lambda_R + \delta$. That is we replace $\mathbf{C}(\Lambda_R)$ by $\mathbf{C}(\Lambda_R{+}\delta)$ or equivalently substitute $\mid \delta\rangle_q$ for $\mid 0\rangle_q$. However as we have not altered the vertex operators their commutators are unaltered and their moments still give us a representation of the <u>same</u> Kac–Moody algebra. What has changed is the gradation of the algebra,

$$\begin{aligned} V(\alpha, z) \mid \beta + \delta\rangle &= e^{i\delta.q}e^{-i\delta.q}V(\alpha, z)e^{i\delta.q} \mid \beta\rangle, \\ &= e^{i\delta.q}V(\alpha, z)z^{\alpha.\delta} \mid \beta\rangle. \end{aligned}$$

Thus the representation of the Kac–Moody algebra on $F \otimes \mathbf{C}(\Lambda_R + \delta)$ given by the moments of $V(\alpha,z)$ and $P^i(z)$ with the derivation d is equivalent to a representation on $F \otimes \mathbf{C}(\Lambda_R)$ given by the moments of $V^\delta(\alpha, z) = e^{i\delta.q}V(\alpha, z)z^{\alpha.\delta}$ and $P^i(z)$ with a new derivation $d^\delta = d + \delta.p + \frac{\delta^2}{2}$. We call $V^\delta(\alpha, z)$ a **shifted vertex operator**. In general the vertex operator will no longer be expandable in integral powers of z,

$$\begin{aligned} V^\delta(\alpha, z) &\equiv \sum_{n\in \mathbf{Z}-\alpha.\delta} V^\delta_n(\alpha)z^{-n}, \\ &\equiv \sum_{n\in\mathbf{Z}} V_n(\alpha)z^{-n+\alpha.\delta}. \end{aligned}$$

But $V_n(\alpha) = t^n \otimes E_\alpha$ therefore $V^\delta(\alpha, z) = \sum_{n\in\mathbf{Z}-\alpha.\delta} (t^{n+\alpha.\delta} \otimes E_\alpha)z^{-n}$ and thus $V^\delta_n(\alpha) \equiv t^{n+\alpha.\delta} \otimes E_\alpha$. Comparing this with (3.26) p 66 we can see that we just have a new non–integral gradation of $g^{(\tau)}$ the Kac–Moody algebra constructed using an inner automorphism. It corresponds to looking at the underlying Lie algebra automorphism in the shifted picture. If we take δ to be of the form $\delta = \frac{1}{N}\sum_{i=1}^{\mathrm{rank}\,g} n_i w_i$, where w_i are the fundamental weights of g then we will have,

$$V^{\delta}(\alpha, z) = \sum_{n \in \frac{1}{N}Z} V_n^{\delta}(\alpha) z^{-n}.$$

The partition function for the Hilbert space of this newly graded representation, $H^{\delta} \equiv F \otimes \mathbf{C}(\Lambda_R + \delta)$ is easily seen to be given by,

$$P_{\delta}(q) = \frac{\sum_{\alpha \in \Lambda_R} q^{\frac{1}{2}(\alpha+\delta)^2}}{\prod_{n=1}^{\infty}(1-q^n)^{\mathrm{rank}\,g}}.$$

In particular the vacuum has a conformal weight of $\frac{\delta^2}{2}$,

$$d^{\delta} \mid 0\rangle = \frac{\delta^2}{2} \mid 0\rangle .$$

Also the vacuum will be degenerate if and only if there are any $\alpha \in \Lambda_R$ for which $(\alpha+\delta)^2 = \delta^2$.

7.2 Determination of the invariant algebras.

We shall now explain how we can determine the invariant, or zero–graded subalgebras of any twisted vertex operator representation of a simply laced Kac–Moody algebra. The main idea behind this calculation is to compare the twisted vertex operator representation with the equivalent shifted vertex operator representation of the same gradation. Once we have established the equivalence then the invariant subalgebra is easily obtained by the application of Theorems (3.2) and (3.4). Really we need to know how the Lie algebra automorphism producing the gradation looks in the two pictures. That is given a root system automorphism $\sigma \in \mathrm{aut}\,\Phi_g$ which we extend to a

Lie algebra automorphism $\Sigma = (\sigma, \psi)$ we want to know what shift vector produces the same automorphism,

$$\sigma \Longleftrightarrow \delta.$$

To establish the correspondence we compute certain characteristics of the automorphism and the corresponding representation in the two pictures. In particular we can calculate the following,

1. Order of Σ, N.
2. Dimension of the invariant subalgebra, $\dim \mathrm{g}_0$.
3. Trace of Σ on the Lie algebra g, $\mathrm{Tr}\,\Sigma$.
4. Conformal weight of the vacuum, η.
5. Degeneracy of the vacuum, c_σ.

We firstly take an explicit automorphism in a given conjugacy class of $\mathrm{aut}\,\Phi_{\mathrm{g}}$ using the classification in [3] and the results of Chapter 4 and calculate them in the twisted picture. We then examine all the possible shift vectors δ to see which one produces the the same results. There are some restrictions on δ that we will explain later. We start by examining how we can calculate the characteristic quantities in the two pictures.

(I) Twisted picture : The extension of the order n automorphism $\sigma \in \mathrm{aut}\,\Phi_{\mathrm{g}}$ to $\Sigma = (\sigma, \psi^\sigma)$ produces the automorphism given in Section 3.3. In particular $\Sigma : \mathrm{g} \to \mathrm{g}$ such that

$$\mathrm{H} \mapsto \sigma(\mathrm{H}). \tag{7.1}$$

$$\mathrm{E}_\alpha \mapsto \psi^\sigma_\alpha \mathrm{E}_{\sigma(\alpha)}, \tag{7.2}$$

where the phases are those of Lemma (3.6), namely

$$\psi^\sigma_\alpha = 1 \qquad \forall \alpha \in \Lambda^0_R.$$

(1) Order :

$$\begin{aligned} \Sigma^p(\mathrm{E}_\alpha) &= \psi^\sigma_\alpha \psi^\sigma_{\sigma(\alpha)} \cdots \psi^\sigma_{\sigma^{n-1}(\alpha)} \mathrm{E}_\alpha, \\ &= (-1)^{\tilde{\alpha}.\alpha} \mathrm{E}_\alpha. \end{aligned} \tag{7.3}$$

where we have used the fact that for a *simply laced algebra* the ψ^σ_α satisfy

$$\psi^\sigma_\alpha \psi^\sigma_\beta = \frac{\varepsilon(\alpha,\beta)}{\varepsilon(\sigma(\alpha),\sigma(\beta))}\psi^\sigma_{\alpha+\beta}$$

and ε is a 2–cocycle satisfying

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = (-1)^{\alpha.\beta}. \tag{7.4}$$

$\big($Recall that $\tilde{\alpha} \equiv \alpha + \sigma(\alpha) + ... + \sigma^{n-1}(\alpha)\big)$. Thus N, the order of Σ, is n or $2n$ depending on whether $\tilde{\alpha}.\alpha \in 2\mathbb{Z}\ \forall\ \alpha \in \Phi_{\mathrm{g}}$ or not.

(2) $\dim \mathrm{g}_0$: Let α be in an orbit under σ of order $\mathrm{M}(\alpha) < n$, where $\mathrm{M}(\alpha)$ has to divide n,

$$\Sigma^{\mathrm{M}(\alpha)}(\mathrm{E}_\alpha) = (-1)^{\frac{\mathrm{M}(\alpha)}{n}\tilde{\alpha}.\alpha}\mathrm{E}_\alpha.$$

We can therefore only form an invariant element from the orbit of E_α if

$$(-1)^{\frac{\mathrm{M}(\alpha)}{n}\tilde{\alpha}.\alpha} = 1.$$

If m is the dimension of the invariant subspace of σ then there are also m invariant elements in the Cartan subalgebra. Thus the dimension of the invariant subalgebra g_0 is given by

$$\dim \mathrm{g}_0 = \left(\text{No. of } \alpha \in \mathrm{O}_\sigma(\Phi_{\mathrm{g}}) \text{ with } \tilde{\alpha}.\alpha \in \frac{2n}{\mathrm{M}(\alpha)}\mathbb{Z}\right) + m, \tag{7.5}$$

where $\mathrm{O}_\sigma(\Phi_{\mathrm{g}})$ is a set of roots such that it contains one and only one from each distinct orbit of Φ_{g} under σ.

For NFPAs of order n we have,

$$\dim \mathrm{g}_0 = \frac{1}{n}|\Phi_{\mathrm{g}}|.$$

(3) Trace :

$$\begin{aligned}\mathrm{Tr}\,\Sigma &= \text{No. of fixed roots under } \sigma + \mathrm{Tr}\,\sigma, \\ &= |\ \Phi^0_{\mathrm{g}}\ | + \mathrm{Tr}\,\sigma.\end{aligned} \tag{7.6}$$

(4) Conformal weight of the vacuum :

$$\eta = \frac{1}{4n^2} \sum_{i=1}^{r} \mathrm{n}_i(n - \mathrm{n}_i). \tag{7.7}$$

where the eigenvalues of σ are given by ω^{n_i} with $0 \leq \mathrm{n}_i < n$ for $i = 1, ..., \mathrm{rank}\,\mathrm{g}$. (See p 114).

(5) Vacuum degeneracy : [25] gives

$$\begin{aligned} c_\sigma &= [\det(1 - \sigma_*)]^{\frac{1}{2}} \frac{\mathrm{vol}(\mathrm{P}_0\Lambda_R)}{\mathrm{vol}(\Lambda_R)} \quad , \\ &= \left| \mathrm{torsion} \left\{ \frac{\Lambda_R}{(1 - \sigma)\Lambda_W} \right\} \right|^{\frac{1}{2}} \quad , \end{aligned} \tag{7.8}$$

where Λ_W denotes the weight lattice of g and σ_* the restriction of σ to the space perpendicular to the invariant subspace, *i.e.* $\sigma_* \equiv (1 - \mathrm{P}_0)\sigma$. See also Section 6.5.

(II) **Shifted picture** : From Corollary (3.3) we may rewrite a general Lie algebra automorphism, with $(\mathrm{g},\tau) \neq (\mathrm{A}_{2\mathrm{n}}, 2)$, as

$$\begin{aligned} \mathrm{H} &\mapsto \mathrm{X}(\mathrm{H}), \\ \mathrm{E}_\alpha &\mapsto \mathrm{e}^{2\pi i \alpha . \delta} \mathrm{E}_{\mathrm{X}(\alpha)}, \end{aligned}$$

where $\delta = \frac{1}{\mathrm{N}} \sum_{i=1}^{\mathrm{rank}\,\mathrm{g}} \mathrm{s}_i w_i$ and the sequence $(\mathrm{s}_1, \mathrm{s}_2, ..., \mathrm{s}_{\mathrm{rank}\,\mathrm{g}})$ is invariant under the permutation of indices corresponding to X *i.e.* $\mathrm{s}_i = \mathrm{s}_{\mathrm{X}(i)}$, $1 \leq i \leq \mathrm{rank}\,\mathrm{g}$.

(1) order : $\mathrm{N} = \tau \sum_{i=0}^{\mathrm{O}(\mathrm{g},\tau)} \mathrm{k}_i^\tau \mathrm{s}_i$, where k_i^τ are the Kac labels for $\mathrm{g}^{(\tau)}$ For an inner automorphism this gives $\mathrm{N} = \tau \sum_{i=0}^{\mathrm{rank}\,\mathrm{g}} \mathrm{k}_i \mathrm{s}_i$, with k_i being the Kac labels of the extended Dynkin diagram of g.

Note : δ defines a unique automorphism of g of order N. As all the $\mathrm{s}_i \geq 0$ δ is dominant, that is it lies in, or along the walls of the fundamental Weyl chamber. Any vector differing from δ by an element of the weight lattice, Λ_W, will produce the same automorphism. However δ is the shortest vector in this equivalence class.

(2) $\dim g_0$: We can form an invariant element from E_α if $\Sigma^\tau(E_\alpha) = E_\alpha$ that is if $e^{2\pi i(\alpha+\ldots+X^\tau(\alpha)).\delta} = 1$. But $\alpha.\delta = \ldots = X^\tau(\alpha).\delta$ so that we require $\tau\alpha.\delta \in \mathbb{Z}$. In this case the invariant element would be

$$\hat{E}_\alpha = E_\alpha + \omega E_{X(\alpha)} + \ldots + \omega^{(\tau-1)} E_{X^{(\tau-1)}(\alpha)} \quad \text{with } \omega \equiv e^{2\pi i\alpha.\delta}.$$

So the dimension of the invariant subalgebra is

$$\dim g_0 = (\text{No. of } \alpha \in O_X(\Phi_g) \text{ with } \tau\alpha.\delta \in \mathbb{Z}) + O(g,\tau)\ , \tag{7.9}$$

which for the special case of an inner automorphism reduces to

$$\dim g_0 = (\text{No. of } \alpha \in \Phi_g \text{ with } \alpha.\delta \in \mathbb{Z}) + \operatorname{rank} g\ . \tag{7.10}$$

(3) Trace :

$$\begin{aligned}\operatorname{Tr}\Sigma &= \sum_{\substack{\alpha\in\Phi_g\\ X(\alpha)=\alpha}} e^{2\pi i\alpha.\delta} + \operatorname{Tr}X\,,\\ &= 2\sum_{\substack{\alpha\in\Phi_g^+\\ X(\alpha)=\alpha}} \cos(2\pi\alpha.\delta) + \operatorname{Tr}X\,.\end{aligned}$$

For an inner automorphism (*i.e.* $X = \mathbb{1}$) this becomes

$$\operatorname{Tr}\Sigma = 2\sum_{\alpha\in\Phi_g^+} \cos(2\pi\alpha.\delta) + \operatorname{rank} g\,. \tag{7.11}$$

(4) Conformal weight of the vacuum : For an **inner automorphism**,

$$\eta = \frac{\delta^2}{2}\,.$$

(5) Vacuum degeneracy : Recall that for an **inner automorphism** the numerator of the partition function is of the form,

$$\sum_{\alpha\in\Lambda_R} q^{\frac{1}{2}(\alpha+\delta)^2}\,.$$

Thus the vacuum degeneracy is given by

$$c_\sigma = |\,\{\alpha \in \Lambda_R \mid (\alpha+\delta)^2 = \delta^2\}\,|\ .$$

The calculation of c_σ in the shifted picture is aided by the following result.

Lemma (7.1) : For inner automorphisms

(1) $c_\sigma \geq 1$ if and only if $s_0 = 0$.

(2) $\alpha \in \{\alpha \in \Lambda_R \mid (\alpha+\delta)^2 = \delta^2\}$ implies that $\alpha^2 \leq \dfrac{\mathrm{rank}\, g}{2}$.

Proof :

(1) If $(\alpha+\delta)^2 = \delta^2$ then

$$-\alpha.\delta = \frac{\alpha^2}{2}\,. \tag{7.12}$$

So $c_\sigma > 1$ iff $\exists\ \alpha \in \Lambda_R$ such that $\alpha.\delta = -1$. But δ is positive so that any such α will be negative. In addition δ is dominant so that for any $\alpha \in \Phi_g^+$ we have $\alpha.\delta \leq \alpha_H.\delta$ and thus

$$-\alpha.\delta \geq -\alpha_H.\delta \ \ \forall \alpha \in \Phi_g^+.$$

In particular $c_\sigma > 1$ iff $\alpha_H.\delta = 1$. Now $\alpha_H = \sum_{i=1}^{\mathrm{rank}\, g} k_i\alpha_i$ thus using the fact that $\alpha_i.w_j = \delta_{ij}$ and the definition of N we have

$$\begin{aligned} \alpha_H.\delta &= \frac{1}{N}\sum_{i=1}^{r} k_i s_i\,, \\ &= 1 - \frac{s_0}{N}\,. \end{aligned} \tag{7.13}$$

Therefore

$$-\alpha.\delta \geq \frac{s_0}{N} - 1 \qquad \forall \alpha \in \Phi_g\,.$$

(2) We have the inequality

$$-\mid\alpha\mid\mid\delta\mid \leq \alpha.\delta \leq \mid\alpha\mid\mid\delta\mid\,,$$

so that (7.12) cannot be satisfied unless $\mid\alpha\mid\mid\delta\mid \geq \frac{1}{2}\alpha^2$, *i.e.* unless

$$\delta^2 \geq \frac{\alpha^2}{4}.$$

By considering the expression for η in the twisted picture we see that

$$\eta \leq \frac{\operatorname{rank} g}{16},$$

with equality only for the automorphism $\alpha \mapsto -\alpha$. Therefore

$$\delta^2 = 2\eta \leq \frac{\operatorname{rank} g}{8},$$

and

$$\alpha^2 \leq \frac{\operatorname{rank} g}{2}.$$

Corollary (7.2) : For all inner automorphisms of Lie algebras with $\operatorname{rank} g \leq 8$ we only need to check negative roots α with $\alpha^2 = 2$ to see if they contribute to the vacuum degeneracy except for the case $\operatorname{rank} g = 8$ and $\alpha \mapsto -\alpha$ when a single vector of the form $\alpha = -2\delta$ with $\alpha^2 = 4$ can contribute.

So for an inner automorphism the vacuum is degenerate iff $s_0 = 0$. In the case that it is and $\operatorname{rank} g \leq 8$, with the above proviso for the automorphism $\alpha \mapsto -\alpha$ in $\operatorname{rank} g = 8$, we can determine the degeneracy by counting the number of length squared two roots for which

$$\alpha.\delta = \alpha_H.\delta .$$

Such roots have the same components as α_H in positions corresponding to non–zero components of δ.

In summary we have the following five restrictions,

$$N = \tau \sum_{i=0}^{O(g,\tau)} k_i^{\tau} s_i ,$$

$$(\text{No. of } \alpha \in O_\sigma(\Phi_g) \text{ with } \tilde{\alpha}.\alpha \in \frac{2n}{M(\alpha)}\mathbb{Z}) + m$$
$$= (\text{No. of } \alpha \in O_X(\Phi_g) \text{ with } \tau\alpha.\delta \in \mathbb{Z}) + O(g,\tau).$$

$$|\,\Phi_g^0\,| + \operatorname{Tr}\sigma = 2 \sum_{\substack{\alpha\in\Phi_g^+ \\ X(\alpha)=\alpha}} \cos(2\pi\alpha.\delta) + \operatorname{Tr} X ,$$

and for **inner** automorphisms,

$$\frac{1}{4n^2}\sum_{i=1}^{r} \mathrm{n}_i(n-\mathrm{n}_i) = \frac{\delta^2}{2},$$

$$\left|\text{torsion}\left\{\frac{\Lambda_R}{(1-\sigma)\Lambda_W}\right\}\right|^{\frac{1}{2}} = |\{\alpha \in \Lambda_R \mid (\alpha+\delta)^2 = \delta^2\}|\,.$$

These are usually enough to calculate δ but very occasionally it is necessary to consider some higher levels in the Kac–Moody gradation. In this case, for inner automorphisms, we can compare the two partition functions,

$$\mathrm{c}_\sigma \frac{\sum\limits_{\alpha\in \mathrm{P}_0\Lambda_R} \mathrm{q}^{\frac{1}{2}\alpha^2+\eta}}{\prod\limits_{\mathrm{n}=1}^{\infty}(1-\mathrm{q}^{\frac{\mathrm{n}}{\mathrm{N}}})^{\mathrm{d}(\mathrm{n}\bmod \mathrm{N})}} = \frac{\sum\limits_{\alpha\in\Lambda_R} \mathrm{q}^{\frac{1}{2}(\alpha+\delta)^2}}{\prod\limits_{\mathrm{n}=1}^{\infty}(1-\mathrm{q}^{\mathrm{n}})^{\mathrm{rank}\,\mathrm{g}}}\,.$$

Note : We can consider σ to be of order $\mathrm{M}n$, $\mathrm{M} \in \mathbb{Z}^+$, rather than of order n without altering the twisted vertex operator construction and in particular the partition function. It is sometimes useful to consider N to be the 'order' of σ [5,19].

We can in fact determine the lowest levels of the spectrum in the shifted picture by systematically considering roots of decreasing height. Recall that a root $\alpha = \sum\limits_{i=1}^{\mathrm{rank}\,\mathrm{g}} \mathrm{n}_i\alpha_i$ has $\mathrm{ht}(\alpha) = \sum\limits_{i=1}^{\mathrm{rank}\,\mathrm{g}} \mathrm{n}_i$. As we also know these levels from the twisted picture, assuming we know c_σ, we can calculate some of the components of δ straight away. The number we can determine varies from algebra to algebra. We define the level of a state to be its eigenvalue under $\mathrm{N}(\mathrm{L}_0-\eta)$. We then label all the states starting from those at the lowest level and working up. Finally we denote the level of state $\mid i\rangle$ by L_i and define

$$\Delta_i \equiv \mathrm{L}_{i+1} - \mathrm{L}_i\,.$$

In the twisted picture the level of a state $\mathrm{c}_{-r_1}.....\mathrm{c}_{-r_m} \mid \alpha_0\rangle$ is

$$\mathrm{N}\left(\frac{\alpha_0^2}{2} + \sum_{i=1}^{m} r_i\right).$$

In the shifted picture the level of a state $c_{-n_1}.....c_{-n_m} \mid \alpha\rangle$ is

$$\mathrm{N}\left(\frac{\alpha^2}{2} + \alpha.\delta + \sum_{i=1}^{m} n_i\right).$$

If $\delta = \frac{1}{\mathrm{N}} \sum_{i=1}^{\mathrm{rank}\,\mathrm{g}} \mathrm{s}_i w_i$ and $\alpha = \sum_{i=1}^{\mathrm{rank}\,\mathrm{g}} \mathrm{n}_i \alpha_i$ then the level of the lowest state $\mid 0\rangle$ is 0 and the level of $\mid \alpha\rangle$, $\alpha^2 = 2$, is $\mathrm{N} + \mathrm{n}_i \mathrm{s}_i$.

Delta is assumed dominant so that $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ implies that $\delta.\alpha \leq \delta.\beta$. Let $\alpha^{(1)}, ...,\alpha^{(m)}$ be the longest sequence of roots of length squared two so that $\mathrm{ht}(\alpha^{(1)}) > \mathrm{ht}(\alpha^{(2)}) > .... > \mathrm{ht}(\alpha^{(m)})$ and there is no other $\beta \in \Phi_{\mathrm{g}}$ such that $\mathrm{ht}(\beta) = \mathrm{ht}(\alpha^{(i)})$ $i = 1, ..., m$. Obviously $\alpha_1 = \alpha_{\mathrm{H}}$, the highest root. We then have

$$\begin{aligned} \Delta_1 &= \mathrm{N}(1 - \alpha_{\mathrm{H}}.\delta) = s_0, \\ \Delta_i &= (\alpha^{(i-1)} - \alpha^{(i)}).\delta \qquad\qquad i = 2, ..., m. \end{aligned}$$

In general we find $\alpha^{(i-1)} - \alpha^{(i)} = \sum_{i=1}^{\mathrm{rank}\,\mathrm{g}} \delta_{ij}\alpha_i$ *i.e.* a simple root α_j for $i = 2, ..., m$, thus $\Delta_i = \delta_{ij}\mathrm{s}_j$. So we can determine m components of s. In fact m corresponds to the number of different values the Kac labels of g can take. If s denotes the sequence $(\mathrm{s}_0, ..., \mathrm{s}_{\mathrm{rank}\,\mathrm{g}})$ then for inner automorphisms of the exceptional simply laced algebras we have

E_6: $m = 3$;

$$\mathrm{s} = (\Delta_1, W, X, \Delta_3, Y, Z, \Delta_2) \text{ with } X + 2Y + Z = N - \sum_{\mathrm{n}=1}^{3} \mathrm{n}\Delta_{\mathrm{n}}\,.$$

E_7: $m = 4$;

$$\mathrm{s} = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, W, X, Y, Z) \text{ with } 3W + 2X + Y + 2Z = N - \sum_{\mathrm{n}=1}^{4} \mathrm{n}\Delta_{\mathrm{n}}\,. \quad (7.14)$$

E_8: $m = 6$;

$$\mathrm{s} = (\Delta_1, X, Y, \Delta_6, \Delta_5, \Delta_4, \Delta_3, \Delta_2, Z) \text{ with } 2X + 4Y + 3Z = N - \sum_{\mathrm{n}=1}^{6} \mathrm{n}\Delta_{\mathrm{n}}\,. \quad (7.15)$$

For D_4 inner automorphisms we have

D_4: $m = 2$;

$$s = (\Delta_1, X, \Delta_2, Y, Z) \text{ with } X + Y + Z = N - \sum_{n=1}^{2} n\Delta_n . \tag{7.16}$$

Example : E_8, Conjugacy class number 81:

N=60, $c_\sigma = 1$

Oscillators: $c_{-\frac{3}{30}}, c_{-\frac{5}{30}}, c_{-\frac{9}{30}}, c_{-\frac{15}{30}}, c_{-\frac{21}{30}}, c_{-\frac{25}{30}}, c_{-\frac{27}{30}}, c_{-1}$.

Invariant lattice: $P_0\Lambda_R = \{ n\alpha \mid n \in \mathbb{Z} \text{ and } \alpha^2 = \frac{1}{2} \}$

Therefore the first few states have levels which are easily calculated from the twisted partition function.

Level	0	6	10	12	15	16	18	...
No of states	1	1	1	1	2	1	2	...
States	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle, \|6\rangle$	$\|7\rangle$	$\|8\rangle, \|9\rangle$	...

Thus

$$(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6) = (6, 4, 2, 3, 0, 1).$$

Further analysis with the aid of (7.15) gives

$$s = (6, 4, 2, 1, 0, 3, 2, 4, 2, 6) .$$

In general we need to know the vacuum degeneracy c_σ in the twisted picture to perform this calculation. However we can try all the possible values of c_σ given in [16], see also p 142, and so determine c_σ at the same time. We must however remember that the values given in [16] are only for inner automorphisms.

The reverse calculation is also possible, that is the determination of the vacuum degeneracy and the low level part of the spectrum from a given δ. For example in E_8 let

$$s = (d_0, d_8, d_7, d_5, d_4, d_3, d_2, d_1, d_6)$$

where we have excluded the two special cases $s = (1,0,0,0,0,0,0,0,0)$ and $s = (0,0,0,0,0,0,0,1,0)$. If $d_0 \neq 0$ then $c_\sigma = 1$ otherwise we let M be the largest integer such that $d_i = 0$ for all $i \leq M$ and then

$$c_\sigma = 1 + M\,,$$

We also have $\Delta_i = d_{i-1}$ for $i = 1$ to 6.

Example : E_8, conjugacy class number 86. $s = (0,0,0,0,1,0,0,0,0)$, $g_0 = A_4^2$, $c_\sigma = 4$. $(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6) = (0,0,0,0,1,0)$.

An easy method to calculate the vacuum degeneracy, or defect, c_σ for inner automorphisms which leave only the origin fixed and do not correspond to exceptional Carter diagrams was pointed out to the author by [50]. Such Carter diagrams can be obtained from the Dynkin diagram of the corresponding Lie algebra by repeated application of the procedure

$$D(g) \rightarrow D\left(g^{(1)}\right) \rightarrow D\left(g^{(1)}\right) - \{\alpha_i\}$$

where $D(g)$ is the Dynkin diagram of g, $D\left(g^{(1)}\right)$ is the extended Dynkin diagram of g and α_i is a node in this extended diagram. The vacuum degeneracy is

$$c_\sigma = \prod_i k_i\,,$$

the product of the Kac labels of the removed spots.

E.g. Conjugacy class no. 84 of E_8. Carter diagram $= A_2^4$.

$$\begin{aligned} D(E_8) \rightarrow D(E_8^{(1)}) \rightarrow D(A_2 + E_6) \quad & \text{gives } 3, \\ D(A_2 + E_6) \rightarrow D(A_2^{(1)} + E_6^{(1)}) \rightarrow D(A_2^4) \quad & \text{gives } 3, \end{aligned}$$

therefore $c_\sigma = 3.3 = 9$.

It is not clear to the author why this calculation works.

The automorphisms which leave nothing fixed and correspond to connected exceptional Carter diagrams with rank g spots ($g(a_i)$ *i.e.* those specifically associated with the algebra g) are *primitive* elements. As we saw on p 144 they have a vacuum degeneracy $c_\sigma = 1$.

7.3 Tables of results.

The results of the calculations for the simply laced exceptional Kac–Moody algebras are given in the following tables. An $*$ indicates when the order of the algebra automorphism Σ is twice that of the corresponding root lattice automorphism σ *i.e.* $N = 2n$. A $\bullet$ denotes a NFPA.

In Table II we denote an outer automorphism that corresponds to $-\mathbb{1}$ times an inner automorphism obtained from a Carter diagram by just the Carter diagram. There is a problem in two of the entries in this table as we have a number of sets of indices s which are indistinguishable by our calculations. This problem would probably vanish if we were able to construct the corresponding γ–shifted vertex operator representation [5] and thus determine values for the vacuum degeneracy and conformal weight.

TABLE I: *25 conjugacy classes of inner automorphisms in* $aut\,\Phi_{E_6}$ *and their corresponding invariant subalgebras.*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
1	ϕ	(0,0,0,0,0,0,0)	E_6	78	1	1	0
2	A_1	(2,0,0,0,0,0,1)	$A_5 \oplus \mathbf{R}$	34	4*	1	$\frac{1}{16}$
3	A_1^2	(2,1,0,0,0,1,0)	$D_4 \oplus \mathbf{R}^2$	14	4*	1	$\frac{1}{8}$
4	A_2	(1,0,0,0,0,0,1)	$A_5 \oplus \mathbf{R}$	15	3	1	$\frac{1}{9}$
5	A_1^3	(1,0,0,1,0,0,0)	$A_2^2 \oplus A_1 \oplus \mathbf{R}$	2	4*	1	$\frac{3}{16}$
6	$A_2 \times A_1$	(4,3,0,0,0,3,1)	$A_3 \oplus \mathbf{R}^3$	7	12*	1	$\frac{25}{144}$
7	A_3	(2,1,0,0,0,1,2)	$A_3 \oplus \mathbf{R}^3$	6	8*	1	$\frac{5}{32}$
8	A_1^4	(0,0,0,0,0,0,1)	$A_5 \oplus A_1$	-2	2	2	$\frac{1}{4}$
9	$A_2 \times A_1^2$	(2,0,1,2,1,0,0)	$A_1^3 \oplus \mathbf{R}^3$	-1	12*	1	$\frac{17}{72}$
10	A_2^2	(1,1,0,0,0,1,0)	$D_4 \oplus \mathbf{R}^2$	6	3	1	$\frac{2}{9}$
11	$A_3 \times A_1$	(2,1,1,0,1,1,0)	$A_2 \oplus \mathbf{R}^4$	2	8*	1	$\frac{7}{32}$
12	A_4	(1,1,0,0,0,1,1)	$A_3 \oplus \mathbf{R}^3$	3	5	1	$\frac{1}{5}$
13	D_4	(1,0,0,1,0,0,1)	$A_2^2 \oplus \mathbf{R}^2$	1	6	1	$\frac{7}{36}$
14	$D_4(a_1)$	(1,0,0,1,0,0,0)	$A_2^2 \oplus A_1 \oplus \mathbf{R}$	2	4	1	$\frac{3}{16}$
15	$A_2^2 \times A_1$	(1,1,0,3,0,1,0)	$A_1^3 \oplus \mathbf{R}^3$	-2	12*	1	$\frac{41}{144}$
16	$A_3 \times A_1^2$	(0,0,1,0,1,0,2)	$A_1^4 \oplus \mathbf{R}^2$	-2	8*	2	$\frac{9}{32}$
17	$A_4 \times A_1$	(3,1,3,1,3,1,0)	$A_1 \oplus \mathbf{R}^5$	-1	20*	1	$\frac{21}{80}$
18	A_5	(2,2,1,0,1,2,1)	$A_1 \oplus \mathbf{R}^5$	2	12*	1	$\frac{35}{144}$
19	D_5	(1,1,0,1,0,1,1)	$A_1^2 \oplus \mathbf{R}^4$	0	8	1	$\frac{15}{64}$
20	$D_5(a_1)$	(4,3,1,2,1,3,2)	$\mathbf{R}^6$	1	24*	1	$\frac{65}{288}$
• 21	A_2^3	(0,0,0,1,0,0,0)	A_2^3	-3	3	3	$\frac{1}{3}$
22	$A_5 \times A_1$	(0,0,1,0,1,0,1)	$A_1^4 \oplus \mathbf{R}^2$	-2	6	2	$\frac{11}{36}$
23	E_6	(1,1,1,1,1,1,1)	$\mathbf{R}^6$	-1	12	1	$\frac{13}{48}$
• 24	$E_6(a_1)$	(1,1,1,0,1,1,1)	$A_1 \oplus \mathbf{R}^5$	0	9	1	$\frac{7}{27}$
25	$E_6(a_2)$	(1,1,0,1,0,1,0)	$A_1^3 \oplus \mathbf{R}^3$	1	6	1	$\frac{1}{4}$

TABLE II: *25 conjugacy classes of outer automorphisms in aut* Φ_{E_6} *and their corresponding invariant subalgebras.*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
• 1	ϕ	(0,1,0,0,0)	C_4	-6	2		$\frac{3}{8}$
2	A_1	(0,0,1,0,0)	$A_3 \oplus A_1$	-2	4*		$\frac{5}{16}$
3	A_1^2	(0,0,0,0,1)	$B_3 \oplus A_1$	2	4*		$\frac{1}{4}$
4	A_2	(0,1,1,0,0)	$A_3 \oplus \mathbf{R}$	-3	6		$\frac{23}{72}$
5	A_1^3	(1,1,0,0,0)	$C_3 \oplus \mathbf{R}$	6	4*		$\frac{3}{16}$
6	$A_2 \times A_1$	(0,0,1,0,2)	$A_1^3 \oplus \mathbf{R}$	1	12*		$\frac{37}{144}$
7	A_3	(0,2,0,0,1)	$B_2 \oplus A_1 \oplus \mathbf{R}$	-2	8*		$\frac{9}{32}$
8	A_1^4	(1,0,0,0,0)	F_4	26	2		$\frac{1}{8}$
9	$A_2 \times A_1^2$	(2,2,0,0,1)	$A_2 \oplus \mathbf{R}^2$	5	12*		$\frac{7}{36}$
10	A_2^2	(0,1,0,0,1)	$B_2 \oplus A_1 \oplus \mathbf{R}$	0	6		$\frac{19}{72}$
11	$A_3 \times A_1$	(1,0,0,1,0)	$A_2 \oplus A_1 \oplus \mathbf{R}$	2	8*		$\frac{7}{32}$
12	A_4	(0,1,1,0,1)	$A_1^2 \oplus \mathbf{R}^2$	-1	10		$\frac{11}{40}$
13	D_4	(1,2,0,0,0)	$C_3 \oplus \mathbf{R}$	-1	6		$\frac{31}{144}$
14	$D_4(a_1)$	(0,0,1,0,0)	$A_3 \oplus A_1$	-2	4		$\frac{5}{16}$
15	$A_2^2 \times A_1$	(2,1,0,1,0)	$A_1^2 \oplus \mathbf{R}^2$	4	12*		$\frac{29}{144}$
16	$A_3 \times A_1^2$	(2,0,0,0,1)	$C_3 \oplus \mathbf{R}$	14	8*		$\frac{5}{32}$
17	$A_4 \times A_1$	(1,2,2,1,0)	$A_1 \oplus \mathbf{R}^3$	3	20*		$\frac{17}{80}$
18	A_5	(1,0,1,1,0)	$A_1^2 \oplus \mathbf{R}^2$	0	12*		$\frac{35}{144}$
19	D_5	†		0	8		$\frac{15}{64}$
20	$D_5(a_1)$	‡	$A_1^2 \oplus \mathbf{R}^2$	-1	24*		$\frac{77}{288}$
• 21	A_2^3	(1,0,1,0,0)	$A_2 \oplus A_1 \oplus \mathbf{R}$	3	6		$\frac{5}{24}$
22	$A_5 \times A_1$	(1,0,0,0,1)	$C_3 \oplus \mathbf{R}$	8	6		$\frac{13}{72}$
23	E_6	(1,1,1,0,1)	$A_1 \oplus \mathbf{R}^3$	1	12		$\frac{11}{48}$
• 24	$E_6(a_1)$	(1,1,1,1,1)	$\mathbf{R}^4$	0	18		$\frac{53}{216}$
25	$E_6(a_2)$	(0,0,0,1,0)	A_2^2	-1	6		$\frac{7}{24}$

† s = (1,1,1,0,0) or (0,0,1,0,1).

‡ s = (0,2,2,1,0), (0,4,1,0,3) or (1,2,0,3,0).

TABLE III: *60 conjugacy classes of W_{E_7} and their corresponding invariant subalgebras.*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
1	ϕ	(0,0,0,0,0,0,0,0)	E_7	133	1	1	0
2	A_1	(2,1,0,0,0,0,0,0)	$D_6 \oplus \mathbf{R}$	65	4*	1	$\frac{1}{16}$
3	A_1^2	(2,0,0,0,0,1,0,0)	$D_5 \oplus A_1 \oplus \mathbf{R}$	28	4*	1	$\frac{1}{8}$
4	A_2	(1,1,0,0,0,0,0,0	$D_6 \oplus \mathbf{R}$	34	3	1	$\frac{1}{9}$
5	$(A_1^3)'$	(0,0,0,0,0,1,0,1)	$E_6 \oplus \mathbf{R}$	25	2	1	$\frac{3}{16}$
6	$(A_1^3)''$	(1,0,1,0,0,0,0,0)	$A_5 \oplus A_1 \oplus \mathbf{R}$	9	4*	1	$\frac{3}{16}$
7	$A_2 \times A_1$	(4,1,0,0,0,3,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^2$	14	12*	1	$\frac{25}{144}$
8	A_3	(2,2,0,0,0,1,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^2$	17	8*	1	$\frac{5}{32}$
9	$(A_1^4)'$	(0,1,0,0,0,0,0,0)	$D_6 \oplus A_1$	5	2	2	$\frac{1}{4}$
10	$(A_1^4)''$	(1,0,0,0,0,0,1,1)	$A_5 \oplus \mathbf{R}^2$	5	4*	1	$\frac{1}{4}$
11	$A_2 \times A_1^2$	(2,0,2,1,0,0,0,0)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^2$	2	12*	1	$\frac{17}{72}$
12	A_2^2	(1,0,0,0,0,1,0,0)	$D_5 \oplus A_1 \oplus \mathbf{R}$	7	3	1	$\frac{2}{9}$
13	$(A_3 \times A_1)'$	(2,0,0,1,0,1,0,0)	$A_2 \oplus A_1^3 \oplus \mathbf{R}^2$	5	8*	1	$\frac{7}{32}$
14	$(A_3 \times A_1)''$	(1,0,0,0,1,2,0,2)	$D_4 \oplus \mathbf{R}^3$	13	8*	1	$\frac{7}{32}$
15	A_4	(1,1,0,0,0,1,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^2$	8	5	1	$\frac{1}{5}$
16	D_4	(1,1,1,0,0,0,0,0)	$A_5 \oplus \mathbf{R}^2$	8	6	1	$\frac{7}{36}$
17	$D_4(a_1)$	(1,0,1,0,0,0,0,0)	$A_5 \oplus A_1 \oplus \mathbf{R}$	9	4	1	$\frac{3}{16}$
18	A_1^5	(0,1,0,0,0,1,0,0)	$D_4 \oplus A_1^2 \oplus \mathbf{R}$	1	4*	2	$\frac{5}{16}$
19	$A_2 \times A_1^3$	(1,0,0,0,0,0,1,2)	$A_5 \oplus \mathbf{R}^2$	-2	6	1	$\frac{43}{144}$
20	$A_2^2 \times A_1$	(0,3,0,0,1,0,0,1)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^2$	-1	12*	1	$\frac{41}{144}$
21	$(A_3 \times A_1^2)'$	(0,2,0,1,0,0,0,0)	$A_3 \oplus A_1^3 \oplus \mathbf{R}$	1	8*	2	$\frac{9}{32}$
22	$(A_3 \times A_1^2)''$	(1,0,1,0,1,0,1,0)	$A_2 \oplus A_1^2 \oplus \mathbf{R}^3$	1	8*	1	$\frac{9}{32}$
23	$A_3 \times A_2$	(4,0,2,1,0,5,0,0)	$A_1^4 \oplus \mathbf{R}^3$	2	24*	1	$\frac{77}{288}$
24	$A_4 \times A_1$	(3,0,1,3,0,1,0,0)	$A_1^4 \oplus \mathbf{R}^3$	0	20*	1	$\frac{21}{80}$
25	$(A_5)'$	(1,0,0,0,1,1,0,1)	$D_4 \oplus \mathbf{R}^3$	7	6	1	$\frac{35}{144}$
26	$(A_5)''$	(2,1,0,1,0,2,0,0)	$A_1^4 \oplus \mathbf{R}^3$	3	12*	1	$\frac{35}{144}$
27	$D_4 \times A_1$	(2,1,0,0,0,1,2,2)	$A_3 \oplus \mathbf{R}^4$	4	12*	1	$\frac{37}{144}$
28	$D_4(a_1) \times A_1$	(1,0,0,0,0,0,1,1)	$A_5 \oplus \mathbf{R}^2$	5	4	1	$\frac{1}{4}$
29	D_5	(1,1,1,0,0,1,0,0)	$A_3 \oplus A_1 \oplus \mathbf{R}^3$	3	8	1	$\frac{15}{64}$
30	$D_5(a_1)$	(4,2,2,1,0,3,0,0)	$A_1^3 \oplus \mathbf{R}^4$	4	24*	1	$\frac{65}{288}$

TABLE III: *(continued)*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
31	A_1^6	(0,0,0,1,0,0,0,0)	$A_3^2 \oplus A_1$	-3	4*	4	$\frac{3}{8}$
32	A_2^3	(0,0,1,0,0,0,0,0)	$A_5 \oplus A_2$	-2	3	3	$\frac{1}{3}$
33	$A_3 \times A_1^{3'}$	(0,1,0,0,0,1,0,2)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^2$	-3	8*	2	$\frac{11}{32}$
34	$A_3 \times A_2 \times A_1$	(1,0,3,0,3,0,1,2)	$A_1^3 \oplus \mathbf{R}^4$	-2	24*	1	$\frac{95}{288}$
35	A_3^2	(0,1,0,0,0,1,0,0)	$D_4 \oplus A_1^2 \oplus \mathbf{R}$	1	4	2	$\frac{5}{16}$
36	$A_4 \times A_2$	(1,0,2,1,0,2,0,0)	$A_1^4 \oplus \mathbf{R}^3$	-1	15	1	$\frac{14}{45}$
37	$(A_5 \times A_1)'$	(0,1,0,1,0,0,0,0)	$A_3 \oplus A_1^3$	-1	6	2	$\frac{11}{36}$
38	$(A_5 \times A_1)''$	(1,0,1,1,1,0,1,0)	$A_1^3 \oplus \mathbf{R}^4$	-1	12*	1	$\frac{11}{36}$
39	A_6	(1,0,0,1,0,1,0,0)	$A_2 \oplus A_1^3 \oplus \mathbf{R}^2$	0	7	1	$\frac{2}{7}$
40	$D_4 \times A_1^2$	(0,2,0,1,0,2,0,0)	$A_1^5 \oplus \mathbf{R}^2$	0	12*	2	$\frac{23}{72}$
41	$D_5 \times A_1$	(1,0,0,1,0,0,1,1)	$A_2^2 \oplus \mathbf{R}^3$	-1	8	1	$\frac{19}{64}$
42	$D_5(a_1) \times A_1$	(3,0,1,2,1,0,3,2)	$A_1^2 \oplus \mathbf{R}^5$	0	24*	1	$\frac{83}{288}$
43	D_6	(2,1,0,1,2,2,1,2)	$A_1 \oplus \mathbf{R}^6$	2	20*	1	$\frac{11}{40}$
44	$D_6(a_1)$	(1,1,0,0,0,1,1,1)	$A_3 \oplus \mathbf{R}^4$	3	8	1	$\frac{17}{64}$
45	$D_6(a_2)$	(2,0,1,0,1,0,2,1)	$A_1^3 \oplus \mathbf{R}^4$	3	12*	1	$\frac{19}{72}$
46	E_6	(1,1,1,1,0,1,0,0)	$A_1^3 \oplus \mathbf{R}^4$	0	12	1	$\frac{13}{48}$
47	$E_6(a_1)$	(1,1,0,1,0,1,0,0)	$A_1^4 \oplus \mathbf{R}^3$	1	9	1	$\frac{7}{27}$
48	$E_6(a_2)$	(1,0,1,0,0,1,0,0)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^2$	2	6	1	$\frac{1}{4}$
• 49	A_1^7	(0,0,0,0,0,0,0,1)	A_7	-7	2	8	$\frac{7}{16}$
50	$A_3^2 \times A_1$	(0,0,0,1,0,0,0,0)	$A_3^2 \oplus A_1$	-3	4	4	$\frac{3}{8}$
51	$A_5 \times A_2$	(0,0,1,0,1,0,0,0)	$A_2^3 \oplus \mathbf{R}$	-2	6	3	$\frac{17}{48}$
52	A_7	(0,1,0,1,0,1,0,0)	$A_1^5 \oplus \mathbf{R}^2$	-1	8	2	$\frac{21}{64}$
53	$D_4 \times A_1^3$	(0,0,0,1,0,0,0,1)	$A_3^2 \oplus \mathbf{R}$	-4	6	4	$\frac{55}{144}$
54	$D_6 \times A_1$	(0,1,0,1,0,1,0,1)	$A_1^4 \oplus \mathbf{R}^3$	-2	10	2	$\frac{27}{80}$
55	$D_6(a_2) \times A_1$	(0,1,0,0,0,1,0,1)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^2$	-1	6	2	$\frac{47}{144}$
56	E_7	(1,1,1,1,1,1,1,1)	$\mathbf{R}^7$	-1	18	1	$\frac{133}{432}$
57	$E_7(a_1)$	(1,1,1,0,1,1,1,1)	$A_1 \oplus \mathbf{R}^6$	0	14	1	$\frac{33}{112}$
58	$E_7(a_2)$	(1,1,0,1,0,1,1,1)	$A_1^2 \oplus \mathbf{R}^5$	0	12	1	$\frac{7}{24}$
59	$E_7(a_3)$	(2,1,2,1,2,3,1,3)	$\mathbf{R}^7$	1	30	1	$\frac{203}{720}$
60	$E_7(a_4)$	(1,0,0,1,0,0,1,0)	$A_2^2 \oplus A_1 \oplus \mathbf{R}^2$	2	6	1	$\frac{13}{48}$

TABLE IV: *112 conjugacy classes of W_{E_8} and their corresponding invariant subalgebras.*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
1	ϕ	(0,0,0,0,0,0,0,0,0)	E_8	248	1	1	0
2	A_1	(2,0,0,0,0,0,0,1,0)	$E_7 \oplus \mathbf{R}$	132	4*	1	$\frac{1}{16}$
3	A_1^2	(2,1,0,0,0,0,0,0,0)	$D_7 \oplus \mathbf{R}$	64	4*	1	$\frac{1}{8}$
4	A_2	(1,0,0,0,0,0,0,1,0)	$E_7 \oplus \mathbf{R}$	77	3	1	$\frac{1}{9}$
5	A_1^3	(1,0,0,0,0,0,1,0,0)	$E_6 \oplus A_1 \oplus \mathbf{R}$	28	4*	1	$\frac{3}{16}$
6	$A_2{\times}A_1$	(4,3,0,0,0,0,0,1,0)	$D_6 \oplus \mathbf{R}^2$	33	12*	1	$\frac{25}{144}$
7	A_3	(2,1,0,0,0,0,0,2,0)	$D_6 \oplus \mathbf{R}^2$	44	8*	1	$\frac{5}{32}$
8	$(A_1^4)'$	(0,0,0,0,0,0,0,1,0)	$E_7 \oplus A_1$	24	2	2	$\frac{1}{4}$
9	$(A_1^4)''$	(1,0,0,0,0,0,0,0,1)	$A_7 \oplus \mathbf{R}$	8	4*	1	$\frac{1}{4}$
10	$A_2{\times}A_1^2$	(2,0,0,0,0,1,2,0,0)	$D_5 \oplus A_1 \oplus \mathbf{R}^2$	13	12*	1	$\frac{17}{72}$
11	A_2^2	(1,1,0,0,0,0,0,0,0)	$D_7 \oplus \mathbf{R}$	14	3	1	$\frac{2}{9}$
12	$A_3{\times}A_1$	(2,1,0,0,0,1,0,0,0)	$D_4 \oplus A_2 \oplus \mathbf{R}^2$	16	8*	1	$\frac{7}{32}$
13	A_4	(1,1,0,0,0,0,0,1,0)	$D_6 \oplus \mathbf{R}^2$	23	5	1	$\frac{1}{5}$
14	D_4	(1,0,0,0,0,0,1,1,0)	$E_6 \oplus \mathbf{R}^2$	27	6	1	$\frac{7}{36}$
15	$D_4(a_1)$	(1,0,0,0,0,0,1,0,0)	$E_6 \oplus A_1 \oplus \mathbf{R}$	28	4	1	$\frac{3}{16}$
16	A_1^5	(0,1,0,0,0,0,0,1,0)	$D_6 \oplus A_1 \oplus \mathbf{R}$	4	4*	2	$\frac{5}{16}$
17	$A_2{\times}A_1^3$	(2,0,1,0,0,0,0,0,2)	$A_5 \oplus A_1 \oplus \mathbf{R}^2$	1	12*	1	$\frac{43}{144}$
18	$A_2^2{\times}A_1$	(1,1,0,0,0,0,3,0,0)	$D_5 \oplus A_1 \oplus \mathbf{R}^2$	6	12*	1	$\frac{41}{144}$
19	$(A_3{\times}A_1^2)'$	(1,0,1,0,0,0,1,0,0)	$A_4 \oplus A_1^2 \oplus \mathbf{R}^2$	4	8*	1	$\frac{9}{32}$
20	$(A_3{\times}A_1^2)''$	(0,0,0,0,0,1,0,2,0)	$D_5 \oplus A_1^2 \oplus \mathbf{R}$	12	8*	2	$\frac{9}{32}$
21	$A_3{\times}A_2$	(4,5,0,0,0,1,2,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^3$	5	24*	1	$\frac{77}{288}$
22	$A_4{\times}A_1$	(3,1,0,0,0,3,1,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^3$	7	20*	1	$\frac{21}{80}$

TABLE IV: *(continued)*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_sigma	η
23	A_5	(2,2,0,0,0,1,0,1,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^3$	10	12*	1	$\frac{35}{144}$
24	$D_4 \times A_1$	(2,1,0,0,0,0,0,1,2)	$A_5 \oplus \mathbf{R}^3$	7	12*	1	$\frac{37}{144}$
25	$D_4(a_1) \times A_1$	(1,0,0,0,0,0,0,0,1)	$A_7 \oplus \mathbf{R}$	8	4	1	$\frac{1}{4}$
26	D_5	(1,1,0,0,0,0,1,1,0)	$D_5 \oplus \mathbf{R}^3$	14	8	1	$\frac{15}{64}$
27	$D_5(a_1)$	(4,3,0,0,0,1,2,2,0)	$D_4 \oplus \mathbf{R}^4$	15	24*	1	$\frac{65}{288}$
28	A_1^6	(0,0,0,0,0,1,0,0,0)	$D_5 \oplus A_3$	0	4*	4	$\frac{3}{8}$
29	$A_2 \times A_1^4$	(0,2,0,0,0,0,0,1,0)	$D_6 \oplus A_1 \oplus \mathbf{R}$	-3	6	2	$\frac{13}{36}$
30	$A_2^2 \times A_1^2$	(1,0,0,0,1,0,0,0,2)	$A_3^2 \oplus \mathbf{R}^2$	-2	12*	1	$\frac{25}{72}$
31	A_2^3	(0,0,0,0,0,0,1,0,0)	$E_6 \oplus A_2$	5	3	3	$\frac{1}{3}$
32	$A_3 \times A_1^3$	(0,0,0,1,0,0,0,1,0)	$A_3 \oplus A_2 \oplus A_1^2 \oplus \mathbf{R}$	0	8*	2	$\frac{11}{32}$
33	$A_3 \times A_2 \times A_1$	(1,0,1,0,2,0,3,0,0)	$A_2 \oplus A_1^3 \oplus \mathbf{R}^3$	1	24*	1	$\frac{95}{288}$
34	$(A_3^2)'$	(0,1,0,0,0,0,0,1,0)	$D_6 \oplus A_1 \oplus \mathbf{R}$	4	4	2	$\frac{5}{16}$
35	$(A_3^2)''$	(1,1,0,0,1,0,0,0,0)	$A_3^2 \oplus \mathbf{R}^2$	0	8*	1	$\frac{5}{16}$
36	$A_4 \times A_1^2$	(2,0,3,0,0,0,1,0,1)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^3$	-1	20*	1	$\frac{13}{40}$
37	$A_4 \times A_2$	(1,2,0,0,0,1,2,0,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^3$	2	15	1	$\frac{14}{45}$
38	$(A_5 \times A_1)'$	(0,0,0,0,0,1,0,1,0)	$D_5 \oplus A_1^2 \oplus \mathbf{R}$	6	6	2	$\frac{11}{36}$
39	$(A_5 \times A_1)''$	(1,0,1,0,0,1,1,0,0)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^3$	2	12*	1	$\frac{11}{36}$
40	A_6	(1,1,0,0,0,1,0,0,0)	$D_4 \oplus A_2 \oplus \mathbf{R}^2$	3	7	1	$\frac{2}{7}$
41	$D_4 \times A_1^2$	(0,2,0,0,0,1,0,2,0)	$D_4 \oplus A_1^2 \oplus \mathbf{R}^2$	3	12*	2	$\frac{23}{72}$
42	$D_4 \times A_2$	(1,1,0,0,0,0,0,0,1)	$A_6 \oplus \mathbf{R}^2$	0	6	1	$\frac{11}{36}$
43	$D_4(a_1) \times A_2$	(2,0,1,0,0,0,0,0,2)	$A_5 \oplus A_1 \oplus \mathbf{R}^2$	1	12	1	$\frac{43}{144}$
44	$D_5 \times A_1$	(1,0,0,0,0,1,0,0,1)	$A_4 \oplus A_2 \oplus \mathbf{R}^2$	2	8	1	$\frac{19}{64}$
45	$D_5(a_1) \times A_1$	(3,0,1,0,0,2,1,0,2)	$A_2 \oplus A_1^2 \oplus \mathbf{R}^4$	3	24*	1	$\frac{83}{288}$
46	D_6	(2,2,1,0,0,0,1,2,1)	$A_3 \oplus \mathbf{R}^5$	5	20*	1	$\frac{11}{40}$
47	$D_6(a_1)$	(1,1,0,0,0,0,0,1,1)	$A_5 \oplus \mathbf{R}^3$	6	8	1	$\frac{17}{64}$
48	$D_6(a_2)$	(2,0,1,0,0,0,1,0,1)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^3$	6	12*	1	$\frac{19}{72}$
49	E_6	(1,1,0,0,0,1,1,1,0)	$D_4 \oplus \mathbf{R}^4$	7	12	1	$\frac{13}{48}$
50	$E_6(a_1)$	(1,1,0,0,0,1,0,1,0)	$D_4 \oplus A_1 \oplus \mathbf{R}^3$	8	9	1	$\frac{7}{27}$
51	$E_6(a_2)$	(1,1,0,0,0,0,1,0,0)	$D_5 \oplus A_1 \oplus \mathbf{R}^2$	9	6	1	$\frac{1}{4}$
52	A_1^7	(0,0,1,0,0,0,0,0,0)	$A_7 \oplus A_1$	-4	4*	8	$\frac{7}{16}$

TABLE IV: *(continued)*

Conjugacy Class	Carter Diagram	s	g_0	Tr Σ	Order of Σ	c_σ	η
53	$A_2^3 \times A_1$	(0,0,0,0,0,0,1,0,3)	$A_5 \oplus A_2 \oplus \mathbf{R}$	-3	12*	3	$\frac{19}{48}$
54	$A_3 \times A_1^4$	(0,2,0,0,0,1,0,0,0)	$D_4 \oplus A_3 \oplus \mathbf{R}$	-4	8*	4	$\frac{13}{32}$
55	$A_3 \times A_2 \times A_1^2$	(0,2,0,3,0,0,0,1,0)	$A_3 \oplus A_1^3 \oplus \mathbf{R}^2$	-3	24*	2	$\frac{113}{288}$
56	$A_3^2 \times A_1$	(0,0,0,0,0,1,0,0,0)	$D_5 \oplus A_3$	0	4	4	$\frac{3}{8}$
57	$A_4 \times A_2 \times A_1$	(1,0,4,0,5,0,3,0,3)	$A_1^4 \oplus \mathbf{R}^4$	-2	60*	1	$\frac{269}{720}$
58	$A_4 \times A_3$	(1,2,0,1,4,0,3,0,0)	$A_1^4 \oplus \mathbf{R}^4$	-1	40*	1	$\frac{57}{160}$
59	$A_5 \times A_1^2$	(0,0,1,1,0,0,0,1,0)	$A_3 \oplus A_1^3 \oplus \mathbf{R}^2$	-2	12*	2	$\frac{53}{144}$
60	$A_5 \times A_2$	(0,0,0,1,0,0,2,0,0)	$A_2^3 \oplus A_1 \oplus \mathbf{R}$	1	12*	3	$\frac{17}{48}$
61	$A_6 \times A_1$	(1,0,3,1,0,0,3,0,0)	$A_2 \oplus A_1^3 \oplus \mathbf{R}^3$	-1	28*	1	$\frac{39}{112}$
62	$(A_7)'$	(0,1,0,0,0,1,0,1,0)	$D_4 \oplus A_1^2 \oplus \mathbf{R}^2$	2	8	2	$\frac{21}{64}$
63	$(A_7)''$	(1,1,0,1,0,1,1,0,0)	$A_1^4 \oplus \mathbf{R}^4$	0	16*	1	$\frac{21}{64}$
64	$D_4 \times A_1^3$	(0,0,1,0,0,2,0,0,0)	$A_3^2 \oplus A_1 \oplus \mathbf{R}$	-1	12*	4	$\frac{55}{144}$
65	$D_4 \times A_3$	(0,2,2,1,0,0,0,3,0)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^3$	-1	24*	2	$\frac{101}{288}$
66	$D_4(a_1) \times A_3$	(0,0,0,1,0,0,0,1,0)	$A_3 \oplus A_2 \oplus A_1^2 \oplus \mathbf{R}$	0	8*	2	$\frac{11}{32}$
67	$D_5 \times A_1^2$	(0,1,1,0,0,0,0,1,0)	$A_5 \oplus A_1 \oplus \mathbf{R}^2$	-2	8	2	$\frac{23}{64}$
68	$D_5 \times A_2$	(1,2,0,0,0,1,2,0,3)	$A_3 \oplus A_1 \oplus \mathbf{R}^4$	-1	24	1	$\frac{199}{576}$
69	$D_5(a_1) \times A_2$	(1,1,0,1,0,0,3,0,2)	$A_2 \oplus A_1^2 \oplus \mathbf{R}^4$	0	24*	1	$\frac{97}{288}$
70	$D_6 \times A_1$	(0,1,0,1,0,2,0,2,0)	$A_1^5 \oplus \mathbf{R}^3$	1	20*	2	$\frac{27}{80}$
71	$D_6(a_2) \times A_1$	(0,1,0,1,0,0,0,2,0)	$A_3 \oplus A_1^3 \oplus \mathbf{R}^2$	2	12*	2	$\frac{47}{144}$
72	$E_6 \times A_1$	(1,0,1,0,0,1,0,0,1)	$A_2^2 \oplus A_1 \oplus \mathbf{R}^3$	-1	12	1	$\frac{1}{3}$
73	$E_6(a_1) \times A_1$	(3,0,3,1,0,3,1,0,0)	$A_1^4 \oplus \mathbf{R}^4$	0	36*	1	$\frac{139}{432}$
74	$E_6(a_2) \times A_1$	(1,0,0,0,1,0,1,0,1)	$A_3 \oplus A_1^2 \oplus \mathbf{R}^3$	1	12*	1	$\frac{5}{16}$
75	D_7	(2,2,1,0,1,1,0,1,1)	$A_1^2 \oplus \mathbf{R}^6$	0	24*	1	$\frac{91}{288}$
76	$D_7(a_1)$	(4,2,1,0,1,3,0,1,3)	$A_1^2 \oplus \mathbf{R}^6$	1	40*	1	$\frac{49}{160}$
77	$D_7(a_2)$	(3,2,0,0,2,1,0,0,1)	$A_2^2 \oplus \mathbf{R}^4$	1	24	1	$\frac{175}{576}$
78	E_7	(2,2,1,0,1,2,2,2,1)	$A_1 \oplus \mathbf{R}^7$	2	36*	1	$\frac{133}{432}$
79	$E_7(a_1)$	(2,2,1,0,1,0,2,2,1)	$A_1^2 \oplus \mathbf{R}^6$	3	28*	1	$\frac{33}{112}$
80	$E_7(a_2)$	(1,1,0,0,0,1,0,1,1)	$A_3 \oplus A_1 \oplus \mathbf{R}^4$	3	12	1	$\frac{7}{24}$
81	$E_7(a_3)$	(6,4,2,1,0,3,2,4,2)	$A_1 \oplus \mathbf{R}^7$	4	60*	1	$\frac{?}{?}$
82	$E_7(a_4)$	(2,0,0,1,0,1,0,0,0)	$A_2^2 \oplus A_1^2 \oplus \mathbf{R}^2$	5	12*	1	$\frac{?}{?}$

TABLE IV: *(continued)*

Conjugacy Class	Carter Diagram	s	$\mathfrak{g}_0$	Tr Σ	Order of Σ	c_σ	η
• 83	A_1^8	(0,1,0,0,0,0,0,0,0)	D_8	-8	2	16	$\frac{1}{2}$
• 84	A_2^4	(0,0,0,0,0,0,0,0,1)	A_8	-4	3	9	$\frac{4}{9}$
85	$A_3^2\times A_1^2$	(0,0,1,0,0,0,0,0,0)	$A_7\oplus A_1$	-4	4	8	$\frac{7}{16}$
• 86	A_4^2	(0,0,0,0,1,0,0,0,0)	A_4^2	-2	5	5	$\frac{2}{5}$
87	$A_5\times A_2\times A_1$	(0,0,0,1,0,0,0,0,0)	$A_5\oplus A_2\oplus A_1$	-3	6	6	$\frac{5}{12}$
88	$A_7\times A_1$	(0,0,1,0,0,1,0,0,0)	$A_3^2\oplus A_1\oplus \mathbf{R}$	-2	8	4	$\frac{25}{64}$
89	A_8	(0,0,0,1,0,0,1,0,0)	$A_2^3\oplus A_1\oplus \mathbf{R}$	-1	9	3	$\frac{10}{27}$
90	$D_4\times A_1^4$	(0,1,1,0,0,0,0,0,0)	$A_7\oplus \mathbf{R}$	-5	6	8	$\frac{4}{9}$
91	D_4^2	(0,1,0,0,0,1,0,0,0)	$D_4\oplus A_3\oplus \mathbf{R}$	-2	6	4	$\frac{7}{18}$
• 92	$D_4(a_1)^2$	(0,0,0,0,0,1,0,0,0)	$D_5\oplus A_3$	0	4	4	$\frac{3}{8}$
93	$D_5(a_1)\times A_3$	(0,0,1,0,0,2,0,0,0)	$A_3^2\oplus A_1\oplus \mathbf{R}$	-1	12	4	$\frac{55}{144}$
94	$D_6\times A_1^2$	(0,1,1,0,0,1,0,0,0)	$A_3^2\oplus \mathbf{R}^2$	-3	10	4	$\frac{2}{5}$
95	D_8	(0,1,0,1,0,1,0,1,0)	$A_1^5\oplus \mathbf{R}^3$	-1	14	2	$\frac{5}{14}$
96	$D_8(a_1)$	(0,0,0,1,0,1,0,1,0)	$A_2\oplus A_1^4\oplus \mathbf{R}^2$	0	12	2	$\frac{25}{72}$
97	$D_8(a_2)$	(0,1,2,1,0,2,0,3,0)	$A_1^4\oplus \mathbf{R}^4$	0	30	2	$\frac{31}{90}$
• 98	$D_8(a_3)$	(0,0,0,1,0,0,0,1,0)	$A_3\oplus A_2\oplus A_1^2\oplus \mathbf{R}$	0	8	2	$\frac{11}{32}$
99	$E_6\times A_2$	(0,0,0,1,0,0,1,0,1)	$A_2^3\oplus \mathbf{R}^2$	-2	12	3	$\frac{55}{144}$
100	$E_6(a_2)\times A_2$	(0,0,0,0,0,0,1,0,1)	$A_5\oplus A_2\oplus \mathbf{R}$	0	6	3	$\frac{13}{36}$
101	$E_7\times A_1$	(0,1,1,1,0,1,0,1,0)	$A_1^4\oplus \mathbf{R}^4$	-2	18	2	$\frac{10}{27}$
102	$E_7(a_2)\times A_1$	(0,1,1,0,0,1,0,1,0)	$A_3\oplus A_1^2\oplus \mathbf{R}^3$	-1	12	2	$\frac{17}{48}$
103	$E_7(a_4)\times A_1$	(0,0,1,0,0,0,0,1,0)	$A_5\oplus A_1^2\oplus \mathbf{R}$	1	6	2	$\frac{1}{3}$
• 104	E_8	(1,1,1,1,1,1,1,1,1)	$\mathbf{R}^8$	-1	30	1	$\frac{31}{90}$
• 105	$E_8(a_1)$	(1,1,1,0,1,1,1,1,1)	$A_1\oplus \mathbf{R}^7$	0	24	1	$\frac{95}{288}$
• 106	$E_8(a_2)$	(1,1,1,0,1,0,1,1,1)	$A_1^2\oplus \mathbf{R}^6$	0	20	1	$\frac{13}{40}$
• 107	$E_8(a_3)$	(1,1,0,1,0,0,1,0,0)	$A_2\oplus A_1^3\oplus \mathbf{R}^3$	0	12	1	$\frac{23}{72}$
108	$E_8(a_4)$	(1,0,1,0,1,0,1,1,1)	$A_1^3\oplus \mathbf{R}^5$	1	18	1	$\frac{17}{54}$
• 109	$E_8(a_5)$	(1,1,0,1,0,1,0,1,0)	$A_1^4\oplus \mathbf{R}^4$	1	15	1	$\frac{14}{45}$
• 110	$E_8(a_6)$	(1,0,0,1,0,0,1,0,0)	$A_2^2\oplus A_1^2\oplus \mathbf{R}^2$	2	10	1	$\frac{3}{10}$
111	$E_8(a_7)$	(1,0,1,0,1,0,0,1,0)	$A_2^2\oplus A_1\oplus \mathbf{R}^3$	2	12	1	$\frac{43}{144}$
• 112	$E_8(a_8)$	(1,0,0,0,1,0,0,0,0)	$A_4\oplus A_3\oplus \mathbf{R}$	4	6	1	$\frac{5}{18}$

8. Some Final Comments.

The main original results of this work have been the determination of all possible cases of NFPA Lie algebra automorphisms and the development of a method of calculating the possible invariant algebras of the twisted vertex operator construction of basic representations of Kac–Moody algebras. Along the way we have given an exposition of a classification theorem for conjugacy classes of the Weyl groups which we extended to cover the full automorphism groups of the root systems. We hope that the drawing together and amplification of various other ideas in this work will also prove useful.

We have not looked at tabulating the invariant subalgebras and properties of twisted vertex operator representations of the Kac–Moody corresponding to classical simply laced Lie algebras, A_n and D_n. In particular due to lack of time we have not looked at the most interesting case of D_4, although now that we have developed the methodology for performing these calculations there is nothing to prevent us from going on to perform such an examination.

We have only considered twisted and shifted vertex operator representations separately. However they can be combined into the more general γ–shifted vertex operator representations of [5]. This construction involves a simultaneous twisting by a root system automorphism, σ, and a shifting of the root lattice Λ_R to the coset $\Lambda_R + \gamma$ in V. In general distinct pairs (σ, γ) give distinct representations but there are times when different pairs produce isomorphic representations. The study of such isomorphisms is an interesting exercise as it can lead to non–trivial character or power series identities. The vacuum degeneracy of such representations is due partially to the shifted and partially to the twisted parts of the construction. In some ways this corresponds to constructing a representation of the twisted algebra and then regrading it by a shift. In the case of inner automorphism we were able to calculate the vacuum degeneracies and conformal weights by using the isomorphism between a twisted vertex operator representation, where these things are difficult to calculate, and a shifted vertex operator representation where they become relatively easy to calculate. In the case of an outer automorphism we are led to look for an isomorphism between our twisted representation and some representation with a standard twist and some shift delta which was related to the particular outer automorphism. It would thus be fruitful to examine these constructions in more detail.

Although we did not look at twisted vertex operator representations of non-simple simply laced algebras in this work the methods employed can be directly extended to cover such constructions. There are however added complications if the Lie algebra contains two identical Lie algebras in its decomposition, such as $E_8 \times E_8$, as then there are added outer automorphisms of of Φ_g corresponding to interchanging the equivalent root sublattices. This would probably allow an alternative formulation of the idea of 'confusion' given in [51]. The method could also be extended to the vertex operator representations of non-simply laced algebras given in [2,47]

In addition we have been looking solely at bosonic strings and representations. It would be interesting to extend our study to twisted fermionic representations of Kac-Moody algebras. Another way of implementing the twisting of the heterotic string is via the fermionic representation of $E_8 \times E_8$ [15,17]. This produces some partition function identities.

There is also the possibility of discovering more exotic representations of the Virasoro algebra such as those given in [52,53].

One of the most interesting unanswered questions, in the field of twisted vertex operator representations, is the construction of intertwining operators of different representations of the same algebra. In string theory the intertwining operators (or twisted string emission vertices) correspond to the vertex operators for emitting twisted strings [54]. There are interesting possibilities when the normal vertex operators and intertwining operators combine together to enhance the symmetry generated by the normal vertex operators alone. This seems to occur in only a few special cases [36]. An example of this is the half twist, $\alpha \mapsto -\alpha$ or $\delta = \frac{1}{2}(1,0,0,0,0,0,0,0,0)$, of E_8 which has so(16) as an invariant subalgebra. In this case the intertwining operators restore the original E_8 symmetry. This mechanism is also involved in the correspondence between the $E_8 \times E_8$ and $\frac{\mathrm{spin}(32)}{Z_2}$ heterotic strings and the construction of the 'moonshine module' [15, 23].

From the string theory point of a view we must remember that there are a number of limitations to the application of our results. Physical considerations put limits on the covering torii of the orbifold. In particular modular invariance of the string theory means that we must take the lattice for toroidal compactification to be self-dual. In other ways our construction is of a very special type as our initial space is a torus. In general this could be a more general manifold such as a Calabi-Yau space. Also we could, whilst retaining the toroidal covering space, look at more general orbifolds obtained by dividing out by a non–abelian group. That is by considering the point group to be non–abelian. There are technical problems in this case as we can no longer use the shifted picture of Lie algebra automorphisms. This is because if we can write two automorphisms in the shifted way, *i.e.* $\Sigma(x) = e^{-i\delta.H} x e^{i\delta.H}$, with respect to the same Cartan subalgebra then they must commute.

In the heterotic string theory when we start compactifying six of the ten initially physical dimensions, as well as the sixteen internal dimensions of the left moving modes, we could adopt the more general asymmetric orbifold construction [55,56,57]. It must be borne in mind that the physical motivation for looking at twisted strings on orbifolds is to obtain a twisted model in four dimensions with a reduced but physically attractive gauge group.

Recently the idea of twisted open strings has been revived [58]. These first appeared in [8] in the construction of off–shell amplitudes of the dual model. They were discussed in more detail in [59].

Finite order Lie algebra automorphisms are also important in the construction of some practical tools in the representation theory of Lie algebras. Work is being done to calculate the characters of these finite order automorphisms in irreducible representations of the Lie groups [60]. These character values allow the determination of information about Lie groups and their representations for groups which is not obtainable from more standard methods.

As stated in the introduction another use of twisted/untwisted vertex operators occurs in the representation theory of finite groups. The task of classifying all the finite simple groups was finally finished in 1981. The proof of this classification extends over 10,000 to 15,000 pages of numerous journals. The resulting groups fall into three classes (see [61] for details); groups of Lie type, alternating groups and 26 sporadic groups. There is, as yet, no uniform description of the sporadic groups but one hope is that lattices and vertex operators may lead to a more unified theory of finite simple groups [23,62]. So far they have been used to construct a 'moonshine module' for the monster (or Friendly Giant), F_1, the largest ($\sim 10^{53}$elements) sporadic group which contains some 20 or 21 of the other 25 sporadic groups, the so called 'happy family', as subgroups [23,24,63,64]. This infinite–dimensional representation involves the Leech

Lattice, the lattice corresponding to the densest sphere packing in 24 dimensions, and a cross–bracket algebra. Beyond his there is a vague hope that the monster may lead to a unique four–dimensional string theory (see for eg [35,65]). Philosophically it would be very nice to link a fundamental area of theoretical physics with such a fundamental area of pure mathematics.

It is clear that twisted vertex operators have a long and interesting future ahead of them in both mathematics and physics.

APPENDIX : **Lie algebra roots in an orthonormal basis.**

Type	Roots		
A_n	$e_i - e_j$	$1 \leq i,\ j \leq n+1$	$i \neq j$
B_n	$\pm e_i \pm e_j$	$1 \leq i,\ j \leq n$	$i \neq j$
	$\pm e_i$	$1 \leq i \leq n$	
C_n	$\pm e_i \pm e_j$	$1 \leq i,\ j \leq n$	$i \neq j$
	$\pm 2e_i$	$1 \leq i \leq n$	
D_n	$\pm e_i \pm e_j$	$1 \leq i,\ j \leq n$	$i \neq j$
E_6	$\pm e_i \pm e_j$	$3 \leq i,\ j \leq 7$	$i \neq j$
	$\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$	$\varepsilon_i = \pm 1,\ \prod_{i=1}^{8} \varepsilon_i = 1$	$\varepsilon_1 = \varepsilon_2 = \varepsilon_8$
E_7	$\pm e_i \pm e_j$	$2 \leq i,\ j \leq 7$	$i \neq j$
	$\pm(e_1 + e_8)$		
	$\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$	$\varepsilon_i = \pm 1,\ \prod_{i=1}^{8} \varepsilon_i = 1$	$\varepsilon_1 = \varepsilon_8$
E_8	$\pm e_i \pm e_j$	$1 \leq i,\ j \leq 8$	$i \neq j$
	$\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$	$\varepsilon_i = \pm 1,\ \prod_{i=1}^{8} \varepsilon_i = 1$	
F_4	$\pm e_i \pm e_j$	$1 \leq i,\ j \leq 4$	$i \neq j$
	$\pm e_i$	$1 \leq i \leq 4$	
	$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$		
G_2	$e_i - e_j$	$1 \leq i,\ j \leq 3$	$i \neq j$
	$(e_i + e_j - 2e_k)$	$1 \leq i,\ j,\ k \leq 3$	$i \neq j \neq k$

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