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The Structure of Amplitudes in N=4 SYM

GODDARD, TIMOTHY,DAVID

The Structure of Amplitudes in $\mathcal{N} = 4$ **SYM**

Timothy David Goddard

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Centre for Particle Theory
Department of Mathematical Sciences
University of Durham

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Dedicated to

Those who have meant the most to me during my time here at Durham University.

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Abstract

The study of amplitudes and related quantities in the $\mathcal{N} = 4$ Super Yang-Mills theory is a subject undergoing rapid evolution at the moment. In this work we present a review of some of the key ideas and concepts which we use to calculate ℓ-loop, n-point amplitudes of varying helicity. We show that performing a restriction on the external data of being in $1 + 1$-dimensions allows remarkably compact expressions to be obtained at both MHV and NMHV levels. We use this data to motivate in $1 + 1$-dimensions remarkably simple formulae for all collinear-limits and ultimately a universal uplifting formula which generates all n-point amplitudes of a particular loop-order and helicity configuration from a small set of lower-loop amplitudes. We also use the mechanism of the correlation function $\leftrightarrow$ amplitude duality to construct the integrand for the five-point amplitude in full four-dimensional kinematics to six-loops in the parity-even sector and five-loops in the parity-odd sector. Finally we consider a rewriting of certain known momentum-twistor amplitudes in terms of bi-twistor, six-dimensional X-variables and dimensionally regularise these equations to match known $\mathcal{O}(\varepsilon)$ results. From this we make some observations about the requirements for this process to be successful in the limited

number of cases where the full $\mathcal{O}(\varepsilon)$ solution is known and provide an ansatz for constructing the terms for more complicated amplitudes.

DECLARATION

ACKNOWLEDGEMENTS

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Introduction

Scattering amplitudes in gauge theories (and gravity) are known to have a significantly simpler underlying structure than that which is implied by the manifestly local construction in Feynman diagrams. One theory which has received much attention in recent years and where these simplifications are particularly striking is the planar $\mathcal{N} = 4$ supersymmetric gauge theory. In fact, it is not unreasonable to expect that the entire S-matrix in this theory can one day be determined from methods with their roots in the integrability of planar $\mathcal{N} = 4$ SYM.

In recent years much progress has been made in the calculation of scattering amplitudes in maximally supersymmetric non-abelian Yang-Mills theory in four-dimensions. In particular, interesting structures enabling new results have been found for the amplitude integrand, both in the planar limit and the full non-planar theory. Perturbative calculations utilising Feynman graphs give complicated results with many cancellations and qualities such as gauge-invariance are emergent-features not manifest term-by-term, additionally there are a vast number of contributing diagrams, making it difficult to construct even the integrand. To evaluate the integrand is, of course, the hardest step - which we shall never attempt directly in this work - but it will naturally be facilitated by finding simple concise forms of the integrands.

There have been three principal methods utilised for the generating of integrands. (Generalised) Unitarity is the most widespread technique [24, 25, 37, 40], here one equates the leading singularities of the amplitude with those of an ansatz - consisting of a sum of independent graphs with arbitrary coefficients - which fixes all freedom. There are several restrictions and criteria as to which

graphs occur in the ansatz: in the planar limit one uses dual conformal invariance [33, 36, 56, 58, 59], whereas in the full non-planar theory one can use the colour-kinematics duality [17, 18]. This technique has been used to obtain the four-point amplitude up to five-loops (planar [19, 21, 29, 31] and non-planar [15, 16, 20, 27]), the five-point amplitude to three-loops [22, 48, 104] and the six-point amplitude to two-loops [28, 43].

Second, one may employ a recursion relation determining higher-loop amplitudes in terms of lower-loop ones [10]. The original BCFW [38, 39] recursion relation involved decomposing higher-point *tree*-level amplitudes into products of three-point amplitudes, but subsequently this technique has been updated to *loop* integrands [10]. The use of these on-shell methods, in particular utilising the machinery of writing things in momentum-twistor variables, yields expressions which are exceptionally compact compared to the (potentially) millions of terms in the same calculation but performed using Feynman graphs. By construction BCFW recursion leads to non-local integrands, i.e. individual terms have poles which are not of $\frac{1}{p^2}$ type. Yet the existence of the Feynman graph method guarantees the cancellation of such spurious singularities in the sum of all terms. It remains a formidable problem though, to find simple local forms for the BCFW output, since the recursion procedure - although vastly more concise than any direct graph calculation - still blooms out considerably at higher-loop order (although much progress has been made in the direction of resolving this issue [11]). At this moment explicit formulae for local integrands utilising this method are limited to MHV n-point amplitudes up to three-loops and NMHV n-point amplitudes as far as two-loops [9, 10].

Third, another less widely known but extremely powerful technique starts from an ansatz, but now fixes the coefficients by implementing the exponentiation of infrared (IR) singularities at the level of the integrand by asserting that the log of the amplitude should have a reduced singularity [23]. This method has been used to obtain the four-point amplitude to seven-loops [23] and has been shown to determine the n-point amplitudes at two- and three-loops for any n [82]. Both this method and generalised unitarity customarily use graphs with local integrands. In addition, the trial graphs used in generalised unitarity methods typically contain only Lorentz products, with any parity-odd structures being in the external variables only.

The work presented here differs from all three of these methods, here we use only proposed dualities with other kinematical objects and building amplitudes from strict symmetry-arguments. In Chapter.2 we review some of the

principal concepts, notation and proposals of the planar-limit of the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory. In particular, in addition to simplifications occurring from working in the planar-limit of $\mathcal{N} = 4$ SYM, we take the additional step of imposing a kinematical restriction on external momenta of scattered states. This corresponds to containing all external momenta to reside in (1+1)-dimensions of the full (3+1)-dimensional Minkowski space (the loop momenta remaining unrestricted). We finish Chapter 2 by reviewing known results from [89, 91], where we know the analytical form of the amplitudes given there simplifies considerably when one restricts to these external kinematics, we can thus consider this restriction as a short-cut towards establishing the underlying integrable structure of the amplitude in full kinematics.

In Chapter 3 we demonstrate how we can extend these kinematical restrictions to the NMHV amplitudes [80] dealing with the problem of infinities arising term-by-term from the spurious poles. We give n-point NMHV one-loop amplitudes explicitly [80] as built purely from symmetry considerations. In Chapter 4 we take inspiration from similarities in the form of the 8-point MHV 2-loop and the 8-point NMHV 1-loop amplitudes and their extensions to higher orders. Using this as motivation, and additionally a better understanding of collinear-limits in (1+1)-dimensions, we propose a universal MHV "uplifting" formula to construct high-n amplitudes at a given loop-order from lower-point ones and a piece that vanishes in all allowed collinear limits [80], we then generalise this to a similar uplifting-formula for any n-point, N^{k}MHV, ℓ-loop amplitude [80].

In Chapter 5 we return to the full (3+1)-dimensional Minkowski space for the external momenta. In this theory it is known that the planar amplitude can be generated from n-point functions of the energy-momentum multiplet of the theory [67, 68, 73, 74] and we explore the form and consequences of this proposed duality. Utilising this duality we extend work done at 4-points where the method was used to construct the 6-loop planar and 4-loop non-planar amplitude [69], to the five-particle amplitude as far as five-loops completely and six-loops in the parity-even sector [6]. In Chapter 6 we explain how to rewrite amplitude integrands from momentum-twistors intoa 6-dimensional embedding which can then be useful to attempt to dimensionally regularize these amplitudes obtaining known results at five- and six-points at one- and two-loops. We then propose an ansatz for the form of the dual-conformally invariant μ^2-terms, for n-point, ℓ-loop MHV amplitudes. We give explicitly our dual-conformally

invariant μ^2-term predictions for the terms at five-point, three-loops as well as four-point, four- and five-loop amplitudes.

Finally in Chapter.7 we present some of our key conclusions as well as potential avenues of further research on the topics discussed.

2

REVIEW OF AMPLITUDES IN 2D KINEMATICS

During this chapter we will begin by introducing the objects of interest namely planar, colour-ordered amplitudes. We will then review the manner in which we take the four-dimensional theory into one where external data is in (1+1)-dimensions, and explore the ways in which these limits effect our results. Alongside this, we will be introducing the necessary technology to analyse and express both our questions and results, such as the collinear limit restrictions, the relevant symmetries of $\mathcal{N} = 4$ SYM, the "Symbol" technology etc. Finally we give a brief example of how all these symmetries, notations and technologies can be used to produce the very simple result of the MHV two-loop remainder function for an arbitrary number of external points in our reduced kinematics, as given in [91].

2.1 AMPLITUDES NOTATION

In this thesis we will entirely be concerned by computing colour-ordered, n-point, N^{k}MHV amplitudes in planar $\mathcal{N} = 4$ SYM theory: $A_{n,k}$.

Planarity and Colour-Ordering

First we concentrate on the decomposition of the amplitudes with regards colour, starting with the 't Hooft limit, where planar diagrams dominate, and colour-ordering those diagrams. The gauge group in QCD is SU(3) however we generalize this to SU(N_c), indeed this makes some of the group theory structure more transparent. Gluons would now carry an adjoint colour-index

$a = 1, \ldots, N_c^2 - 1$, whereas quarks and anti-quarks carry an N_c or $\overline{N_c}$ index $i, \bar{j} \in 1, \ldots, N_c$. The generators of $\mathrm{SU}(N_c)$ in the fundamental representation are $N_c \times N_c$ matrices we call $(T^a)_i^{\bar{j}}$.

For each gluon-quark-quark vertex in a generic Feynman diagram we obtain a factor of $(T^a)_i^{\bar{j}}$. For any pure-gluon 3-vertex we include a structure constant f^{abc}, defined by $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ and for each pure-gluon 4-vertex contracted pairs of structure constants $f^{abe} f^{cde}$. We note that clearly all gauge-indices should be contracted and we consider the possible contractions of two generators.

$$(T^a)_{i_1}^{\bar{j}_1}(T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c} \delta_{i_1}^{\bar{j}_1} \delta_{i_2}^{\bar{j}_2} \tag{2.1.1}$$

where the sum over a is implicit. We see that the first term is a single trace over both generators whereas the second term which is suppressed by a power of N_c is a trace over each term separately. In the so-called t' Hooft limit we send $N_c \rightarrow \infty$, suppressing this second type of contraction. The effect this has is that the dominant terms are all of a single trace structure, which in turn suppresses all non-planar diagrams which by nature of their non-planarity have higher trace-structures. For further details on the 't Hooft limit, see the original paper [105].

The colour dependence of the amplitude can be factorised from the kinematic dependence and it is this factorisation which gives us so-called "colour-ordered" amplitudes. To "colour-order" our amplitude is to write the full amplitude in a particular colour decomposition which we here demonstrate at tree level:

$$\mathcal{A}_n^{\mathrm{tree}}\left(\{p_i, \lambda_i, a_i\}\right) = g^{n-2} \sum_{\sigma \in \frac{S_n}{Z_n}} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) A_n^{\mathrm{tree}}\left(\sigma(1^{\lambda_1}), \cdots, \sigma(n^{\lambda_n})\right)$$

$$\tag{2.1.2}$$

here g is the gauge-coupling $\left(\frac{g^2}{4\pi} = \alpha_s\right)$ with p_i and λ_i representing the gluon momenta and helicities respectively and $A_n^{\mathrm{tree}}\left(\sigma(1^{\lambda_1}), \cdots, \sigma(n^{\lambda_n})\right)$ being the "partial amplitudes", which contain all the kinematic information. The sum across $\frac{S_n}{Z_n}$ is required in order to sum over all distinct cyclic orderings in the trace, it denotes the set of all permutations of the n-points however with only one representative from all cyclically equivalent orderings. Note that such a step would not be valid at the non-planar level where we have double-trace structures etc. this represents (mathematically) the fact that in the non-planar picture there is no longer a clear cyclic ordering of external particles. The partial amplitudes are significantly simpler than the full amplitude as they are

now colour-ordered: they receive contributions from diagrams with a particular cyclic ordering of the gluons. Further details can be found in [14, 30, 49, 97, 98]

Supermultiplet and super-amplitude

The expression N^kMHV relates to the helicity-degrees of the external variables. It can be seen from supersymmetric Ward identities [49, 58] that any amplitude with all external particles being gluons with the same helicity or with only a single opposite helicity gluon, give zero:

$$A_n(1^\pm, 2^\pm, \cdots, n^\pm) = 0 = A_n(1^\mp, 2^\pm, 3^\pm, \cdots, n^\pm) \qquad (2.1.3)$$

As such, the first non-zero amplitude is one with $(n - 2)$ positive helicity gluons and 2 negative-helicity gluons, this is called the Maximally Helicity Violating amplitude (MHV). The amplitude with 3 negative helicity gluons and the remainder being positive helicity gluons is called the next-to-MHV (NMHV) amplitude etc. The final non-zero amplitude we can find would be one with only two positive helicity gluons and the rest being negative-helicity, namely $\text{N}^{n-4}\text{MHV} = \overline{\text{MHV}}$ which is also called anti-MHV and is clearly parity-conjugate to the MHV amplitude where we interchange positive and negative helicities.

Each N^kMHV amplitude $A_{n,k}$ is a combination of all possible physical amplitudes involving k+2 negative-helicity gluons and the rest positive-gluons, together with amplitudes related to these by supersymmetry. For example, an amplitude with 4 scalars and the remainder positive-helicity gluons is super-symmetrically related to the MHV amplitude, or an amplitude with 4 negative-helicity fermions and otherwise positive-helicity gluons to the NMHV amplitude etc. These amplitudes are dependent on the bosonic variables which come from on-shell momenta p_i of external particles and fermionic Grassmann four-vectors η_i^A necessary to specify all the particle states of the SYM multiplet:

$$G^+ + \eta_i^A \psi_{i,A} + \eta_i^A \eta_i^B \phi_{AB} + \varepsilon_{ABCD} \eta_i^A \eta_i^B \eta_i^C \bar{\psi}_i^D + \varepsilon_{ABCD} \eta_i^A \eta_i^B \eta_i^C \eta_i^D G^- \quad (2.1.4)$$

Where we define G^+ as a positive-helicity gluon, $\psi_{i,A}$ a positive-helicity fermion etc. Each $A_{n,k}$ is of degree η^{8+4k} where wewould expand A_n as a Taylor polynomial in the Grassmann variables, see [79, 101] for further details.

All $A_{n,k}$ amplitudes can be collected together into a single object which we call a super-amplitude

$$A_n = \sum_{k=0}^{n-4} A_{n,k} \qquad (2.1.5)$$

and each $A_{n,k}$ can be recovered as a term with coefficient of η^{8+4k} in the Taylor expansion in terms of Grassmann variables η_i^A. This means that we can consider the process of scattering amplitudes to be enacted by complete supermultiplets rather than merely by certain combinations of their constituents, the particular components being extracted later from the complete answer as a single term multiplied by a specific Grassmann product.

The Remainder Function

In general, it is customary to factorise out the tree-level contribution A_n^{tree}, as well as the infrared (IR) divergences coming from loops from the other kinematical dependence,

$$A_n = A_n^{\text{tree}} M_n^{\text{BDS}} R_n \qquad (2.1.6)$$

M_n^{BDS} denotes the known BDS-expression [29] which is factorised out as it contains all IR divergences of the amplitude and is known to factorise correctly under simple collinear limits, i.e. where two or more consecutive momenta become collinear. R_n is called the "remainder function".

Using this, we see that to determine the complete amplitude it is sufficient to calculate the remainder function. R_n is a super-function (i.e. a supersymmetric function of both bosonic and fermionic variables) and can be Taylor expanded in Grassmann η's to give N^{k}MHV remainder functions,

$$R_n = \sum_{k=0}^{n-4} R_{n,k} \qquad (2.1.7)$$

where any $R_{n,k}$ is a finite, regularisation-independent, dual-invariant quantity. By 'dual-invariant' we mean that this function is invariant under both the standard conformal operators, and the conformal operators which come from the duality between Wilson loops and amplitudes. To be explicit, in the MHV case $k = 0$, it is predicted that $R_{n,0}$ is dual-conformally invariant and depends on external momenta only through conformal-invariant variables called conformal cross-ratios '$u_{i,j;k,l}$' [58], which we explore in greater detail later. For general N^{k}MHV amplitudes, the dual super-conformal invariance [59] fully present at tree-level [33] becomes partially broken at loop-level, implying $R_{n,k}$ depends

on the external kinematics (momenta and helicities) through the cross-ratios as well as dual super-conformal invariants [59] involving Grassmann variables.

There are conjectured dualities between the super-amplitude, super-Wilson loops [44, 99] and supersymmetric correlation functions [67, 68, 73], either of which is sufficient to explain the presence of the dual superconformal symmetry.

It will be advantageous to consider the logarithm of the remainder super-amplitude, $\mathcal{R}_n = \log(R_n)$. In perturbation theory it can be expanded in powers of the coupling, and independently of this also in powers of η

$$\mathcal{R}_n := \log(R_n) = \sum_{\ell=1}^{\infty} a^\ell \mathcal{R}_n^{(\ell)} = \sum_{k=0}^{n-4} \sum_{\ell=1}^{\infty} a^\ell \mathcal{R}_{n,k}^{(\ell)} \qquad (2.1.8)$$

Where here a is merely the original coupling constant, after expanding out the Taylor series we collect all terms with a prefactor of a and that we call R_1, all terms proportional to a^2 are collected to become R_2 etc. We note here that $\mathcal{R}_{n,k}$ will naturally have contributions from $R_{n,k}$ but additionally from products $R_{n,k'} R_{n,k-k'}$. We note that in our definition of R_n in (2.1.6) we factorised out the entire tree-level super-amplitude (rather than, for example, only the MHV expression as is sometimes done in the literature, see e.g. [46]). Thus all tree-level contributions are cleanly separated from loops and the expansion on the right-hand side of (2.1.8) starts from $\ell = 1$ loops.

For MHV contributions, the expansion starts at 2-loops since $R_{n,0}^{(1)} = 0$, however we will return to this point later on in this chapter when we consider our reduced kinematics and demonstrate this result there with additional restrictions. In general four-dimensional kinematics, non-trivial two-loop contributions start at 6-points, and $R_{6,0}^{(2)}$ was obtained numerically in [28, 57] and later analytic expressions for $R_{6,0}^{(2)}$ were derived in [63, 64, 85]. The result for general n $\left(R_{n,0}^{(2)} \right)$ can be obtained numerically from the algorithm constructed in [7]. The symbol [85] (see Sect 2.6) of the n-point amplitude $R_{n,0}^{(2)}$ is known [45], as is the symbol of the six-point 3-loop MHV amplitude $R_{6,0}^{(3)}$ [51], the six-point, 4-loop, MHV amplitude $R_{6,0}^{(4)}$ [50], the seven-point, 2-loop, MHV amplitude $R_{7,0}^{(2)}$ [83] and the 6-point, 2-loop, NMHV amplitude $R_{6,1}^{(2)}$ [52]. In special two-dimensional kinematics (see Sect 2.2), remarkably concise analytic expressions for $R_{n,0}^{(2)}$ were derived in [62] at n=8 and in [89] for all n. An ansatz for analytic expressions of the three-loop MHV expression $R_{n,0}^{(3)}$ were obtained in [91] for $n = 8$ in special 2d kinematics and further generalised to $n = 10$. This was then completed with additional 'mixed' terms up to $n = 12$ in [47].

2.2 Special Kinematics and Collinear Limits

In the next few sections we follow our work in [80] and draw on earlier work to outline the special two-dimensional kinematics we will subsequently use to explore the structure of the amplitude. Alongside this, in Sect 2.4 we show work presented in [80] on the way in which one can perform collinear limits in 2d kinematics on the remainder function R_n and the manner in which this helps us to constrain the form of our amplitude. These insights will eventually lead to the uplifting formula in Chapter 4.

Variables

Super-amplitudes are functions of bosonic variables (the lightlike momenta p_i of external particles) as well as fermionic variables η_i^A [101] (Here A is an index for the four components of the Grassmann vector, and i labels which variable it belongs to) which take into account the different states in the super Yang-Mills multiplet which are being scattered. All k-components of the N^kMHV amplitudes $A_{n,k}$ arise from the Taylor expansion of the super-amplitude in terms of the Grassmann variables η_i^A (see Sect 5 of [79]).

It will be expedient to rewrite the external data $\{p_i^\mu, \eta_i^A\}$ in terms of "region momenta" $x_{i,i+1}^{\alpha,\dot\alpha}$ and their fermionic counterparts $\theta_i^{\alpha A}$ which are defined as follows

$$p_\mu \sigma_{\alpha\dot\alpha}^\mu := p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} \qquad \alpha, \dot\alpha = 1, 2$$
$$\lambda_i^\alpha \eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} \qquad A = 1, 2, 3, 4 \qquad (2.2.1)$$

λ_i^α and $\tilde\lambda_i^{\dot\alpha}$ are standard two-component helicity spinors in terms of which one can write any massless 4-momenta (p_i). These new chiral superspace coordinates $X_i = (x_i, \theta_i^A)$ define the vertices of the n-sided null polygonal contour for the Wilson loop dual to the n-point super-amplitude [4, 34, 59].

We will also make use of momentum supertwistors [92, 100] which transform linearly under $SU(2,2 \mid 4)$ dual-superconformal transformations. The supertwistors are defined via

$$\mathcal{Z}_i = \left(Z_i^a, \chi_i^A \right) = \left(\lambda_i^\alpha, x_{\alpha\dot\alpha,i} \lambda_i^\alpha; \theta_{\alpha,i}^A \lambda_i^\alpha \right) \qquad (2.2.2)$$

where Z_a denote the four bosonic, and χ^A the four fermionic components.

2.3   REVIEW OF HELICITY-PRESERVING COLLINEAR LIMITS

A collinear limit occurs when a number of consecutive external momenta are changed towards having the same direction i.e. they become 'collinear'. For us the usefullness of these limits will be in using them to restrict the allowable forms of an amplitude. As an example an n-point amplitude must become the $(n-1)$-point amplitude under ANY and ALL collinear limits of two neighbouring external momenta, potentially with a small amount remaining which we will call a 'splitting function' and whose form is highly non-trivial itself. This restriction will later be used extensively to use the form of lower-point amplitudes to build a higher-point amplitude.

Here we will describe how the collinear limits, where $(m+1)$ consecutive momenta ($m \geq 1$) become collinear, act on the remainder super-amplitude R_n and its logarithm $\mathcal{R}_n$. In [80] we found a new and very simple formula containing all collinear limits on all amplitudes in a single simple formula. We begin with the simple and known result that the full super-amplitude A_n factorizes in the $(m+1)$-collinear limit as follows

$$A_n \to A_{n-m} \times \mathrm{Split}_m \qquad\qquad (2.3.1)$$

where A_{n-m} is a super-amplitude with $n-m$ external states, and the expression Split_m denotes the splitting function. The splitting function can on one hand be thought of simply as "everything left over under the collinear limits once the lower-point amplitude has been reformed". However, in fact, the splitting function is itself a super-amplitude, with all the necessary structure which that implies. As such we can stratify Split_m in terms of the helicity-type of those amplitudes in analog with (2.1.7), i.e. $\mathrm{Split}_m = \sum_{p=0}^{k} \mathrm{Split}_{m,p}$. As such, we have different 'types' of splitting functions based on the value of p in this sum. When $p=0$ we call these "helicity-preserving" or "k-preserving" collinear limits, since the helicity form of the principal amplitude remains unchanged (e.g. NMHV $\to$ NMHV), when $p \neq 0$ we call these "helicity-changing" or "k-changing" collinear limits as the principal amplitude changes helicity under the limit (e.g. NMHV $\to$ MHV). If we collect all these different collinear limits

together we can express the effect of a collinear limit of m-points as:

$$A_{n,k} \longrightarrow A_{n-m,k} \times \text{Split}_{m,0} + A_{n-m,k-1} \times \text{Split}_{m,1} + \ldots$$

$$= \sum_{p=0}^{k} A_{n-m,k-p} \times \text{Split}_{m,p} \qquad (2.3.2)$$

where k and p both denote the helicity-configuration of their respective amplitudes $p = 0$ being an MHV amplitude since it has no additional negative gluons, $p = 1$ being an NMHV amplitude as it has one additional negative gluon etc. These are all in analog to standard Grassmann expansion of super-amplitudes (2.1.7).

The simplest collinear limit occurs when just two consecutive momenta in the colour-ordered amplitude become collinear. The amplitude A_n factorizes in this limit into the amplitude with $n-1$ external particles multiplied by the splitting function, $A_n \to A_{n-1} \times \text{Split}_1$. It has been shown [29, 95] that the BDS expression together with the tree-level amplitude, fully account for the splitting amplitude Split_1. This means that when we look at the action of this minimal collinear limit on the remainder super-amplitude it has a particularly simple form, $R_n \to R_{n-1}$ (i.e. there is nothing left over in remainder function after this collinear limit other than the super-amplitude R_{n-1}).

Let us next consider the triple collinear limit where $m+1 = 3$ consecutive momenta become collinear, and furthermore we require that the helicity of the amplitude is conserved, helicity-preserving collinear limits which focus on the $p = 0$ term on the right-hand side of (2.3.2). The new feature of the triple collinear limit compared to the simple collinear limit above, is that the corresponding splitting function is no longer fully accounted for by the BDS expression M^{BDS}. When interpreted in terms of the remainder amplitude, the factorisation theorem for the helicity-preserving triple collinear limit gives

$$\lim_{k \text{ fixed}} R_{n,k} \to R_{n-2,k} \times \text{Split}_{2,0} = R_{n-2,k} \times R_{6,0} \qquad (2.3.3)$$

where $\text{Split}_{2,0}$ is the helicity-preserving triple collinear splitting amplitude (or to be more precise, the part which is not accounted for by the BDS expression). Importantly, this splitting amplitude agrees with the 6-point MHV remainder amplitude $R_{6,0}$ [7, 28].

Finally for helicity-preserving multi-collinear limits with $(m+1)$-collinear momenta, we have

$$\lim_{\substack{k \text{ fixed}}} R_{n,k} \to R_{n-m,k} \times R_{m+4,0} \tag{2.3.4}$$

where similarly to (2.3.3) the splitting amplitude becomes the remainder amplitude $R_{m+4,0}$ itself [89, 90].

2.4   FULL HELICITY CHANGING AND PRESERVING COLLINEAR LIMITS

We are now ready to consider the general multi-collinear case, where we no longer impose any restrictions on preserving the helicity-degree k of the amplitude. We first published this work in [80] and its suprisingly compact form was very important in making it of use for our work there on the Uplifting Formula we derive in Chapter 4. Indeed the following equations can be seen as one of the most important insights which led to the development and implementation of the Uplifting Formula as it made possible the use of collinear-limits as a tool to easily restrict the allowable forms of the remainder functions. We claim the following simple formula as the analog of the super-amplitude factorisation (2.3.1), directly for the remainder super-amplitude

$$R_n \to R_{n-m} \times R_{m+4} \tag{2.4.1}$$

This formula can also be expanded in terms of N^{k-p}MHV components similarly to (2.3.2) except that now all the splitting-function contributions are expressed in terms of R's:

$$\begin{aligned} R_{n,k} \to & R_{n-m,k} \times R_{m,0} + R_{n-m,k-1} \times R_{m,1} + ... \\ = & \sum_{p=0}^{k} R_{n-m,k-p} \times R_{m,p} \end{aligned} \tag{2.4.2}$$

The k-preserving collinear limit (2.3.4) is a special case of these general relations which corresponds to a single term on the right-hand side of (2.4.2) ($p=0$).

The proof of this collinear factorization for R_n in (2.4.1) uses known universal collinear factorisation properties of amplitudes, combined with the dual-superconformal symmetry of R_n. We know that the super-amplitude A_n has

universal collinear factorisation limits (2.3.1), as does M_{BDS} being the exponent of the one-loop MHV amplitude. Therefore the remainder amplitude R_n as defined in (2.1.6) must also have universal factorization properties. Thus we only need to discover what the corresponding splitting super-amplitude is. To do this let us focus on the maximal multi-collinear limits where $n = m + 4$. In this limit from universal factorisation we have $R_{m+4} \to R_4 \times \text{Split}_m = \text{Split}_m$ since R_4 is trivial. On the other hand, the same $(m+1)$-collinear limit can be achieved via a superconformal transformation on all $m + 4$ points which we shall show below (Sect 2.5), therefore we have $R_{m+4} \to R_{m+4}$ in this case. The conclusion is that the splitting amplitude $\text{Split}_m = R_{m+4}$ and (2.4.1) follows.

Taking the logarithm we get a linear realisation of multi-collinear limits,

$$\mathcal{R}_n \to \mathcal{R}_{n-m} + \mathcal{R}_{m+4} \tag{2.4.3}$$

equations (2.4.3) or (2.4.1) constitute our main result as far as general collinear limits are concerned, and they will play a key role in constructing the uplift to general n of the amplitude in the 2d external kinematics which we later turn our attention to. However, first we wish to re-emphasize that ultimately we claim that the simplest, linear realisation of these multi-collinear limits is found by taking the logarithm of the super-amplitude and not acting on the super-amplitude itself. The simplification of collinear limits which we here present is wonderfully compact and allows us a very easy way, when writing our amplitudes in the correct form, to see that amplitudes obey the correct collinear limits. The key upshot, as will be seen in later chapters is that this allows easy use of collinear limit restrictions to constrain the allowed form for amplitudes.

2.5 Collinear limits and (super)conformal transformations

The reason for the very simple form of the collinear factorisation of reduced amplitudes under the $(m+1)$-collinear limit comes from universal collinear factorisation of super-amplitudes, combined with (dual) superconformal symmetry. Applying the $(m+1)$-collinear limit on a $(m+4)$-point reduced amplitude gives the 4-point super-amplitude (which is simply 1 for the reduced super-amplitude) multiplied by the splitting super-amplitude. On the other hand as we shall show now, performing the $(m+1)$-collinear limit on the $(m+4)$-point

super-amplitude can be achieved via a superconformal transformation. Indeed this superconformal transformation will become the definition of the collinear limit, defining precisely the relative speed with which the fermionic coordinates approach collinearity compared to the bosonic variables.

We will give collinear limits in terms of superconformal transformations for the case of interest, namely in 2d kinematics, since the discussion is particularly simple here and the motivation for the simplifications resulting from these kinematics is given in Sect.2.7. We discuss the superconformal group $SL(2|2)$ acting on unconstrained variables (super-twistors) (z, χ) as defined in (2.2.1) and (2.2.2). The bosonic case is simply the well-known Möbius transformation. The general 4d bosonic case was discussed in [3] where it was related to the family of conformal transformations preserving a light-like square and the generalisation of this to the superspace case should follow.

So we begin with an $(m+4)$-point reduced super-amplitude $\mathcal{R}_m(\mathcal{Z}_1, \dots \mathcal{Z}_{m+4})$ (where this is a function of supertwistors $\mathcal{Z}$ defined in (2.2.1) and (2.2.2),) and we wish to perform the $(m+1)$-collinear limit on this. To this effect we want to send $z_{m+4}, z_{m+2}, \dots z_6 \to z_4$ and similarly $\chi_{m+4}, \chi_{m+2}, \dots \chi_6 \to \chi_4$. In particular all odd-point variables are unchanged and we do not act on them (in 2d kinematics they are acted on via a separate $SL(2 \mid 2)_+$ which we can choose to be the identity) but more importantly z_2 and χ_2 are also unchanged. In other words we wish to find an $SL(2 \mid 2)_-$ transformation (or more precisely family of transformations) which keeps z_2, χ_2 fixed whilst all other $z \to z_4$ and all other $\chi \to \chi_4$.

We can find precisely such a transformation. We use standard coset techniques to implement the $SL(2|2)$ transformations. For example, the conformal part of $SL(2|2)$ acts as follows

$$z \to \frac{az + b}{cz + d}, \qquad \chi \to \frac{\chi}{cz + d} \ . \tag{2.5.1}$$

We first use this to send $z_2 \to 0$, $z_4 \to \infty$ and $\chi_2, \chi_4 \to 0$. At this point there is a simple family of transformations keeping these points fixed ($b = c = 0$, $d = 1/a$), so that $z \to a^2 z$, $\chi \to a\chi$ with a parametrising a family of conformal transformations, and $a \to 0$ corresponding to the collinear limit. Finally, transforming back to the original coordinates we thus construct the

explicit conformal transformation implementing our collinear limit as

$$z \to \frac{z_2\, a^2(z-z_4)-z_4(z-z_2)}{a^2(z-z_4)-(z-z_2)}$$

$$\chi \to \frac{a\,\chi\,(z_4-z_2) + (1-a)\,[\,a\,\chi_2\,(z-z_4) + \chi_4\,(z-z_2)]}{(z-z_2)-a^2(z-z_4)}\ . \qquad (2.5.2)$$

Notice that the z transformation is simply a Möbius transformation as expected. The points (z_2, χ_2) and (z_4, χ_4) are fixed, but in the limit $a \to 0$ all other points approach (z_4, χ_4) corresponding to the collinear limit.

In particular when z is close to z_4 the transformation simplifies to

$$z-z_4 \to a^2(z-z_4) + O(z-z_4)^2 \qquad \chi-\chi_4 \to a\,(\chi-\chi_4) + O(z-z_4)\ . \qquad (2.5.3)$$

We see that we are taking a very specific collinear limit, where the χ's approach the limit at half the rate that the z's approach the limit.

Thus we have shown that the $(m+1)$-collinear limit $z_m, z_{m-2}, \ldots z_6 \to z_4$ and similarly $\chi_m, \chi_{m-2}, \ldots \chi_6 \to \chi_4$ can be implemented (and indeed explicitly defined) via a family of superconformal transformations. Since R_{m+4} is superconformally invariant, the function is unchanged by the collinear limit, in particular it is finite and we have $R_{m+4} \to R_{m+4}$. Thus R_{m+4} is the $(m+1)$-collinear splitting amplitude.

2.6 THE SYMBOL

We follow the discussion of [91] and introduce the fundamental concepts associated with the use of "the symbol". There is a wealth of material on the symbol and its role in fundamental physics and scattering amplitudes, for further information see [41, 44, 51, 63, 64, 84, 85]. Essentially this prescription is a method to map highly complicated polylogarithmic functions and their relatives to tensors involving rational functions. By utilising this tool one can render obscure polylogarithmic identities to manifest algebraic identities which the tensor satisfies. Using this the authors of [85] reduced a 17-page formula expressing the direct computation of the hexagon Wilson loop at 2-loops [63, 64] to a single line. Some more recent results which concern amplitudes at the integral level have been given as symbols rather than functions [44, 51].

The "symbol" itself is still a relatively new (at least to physicists') and increasingly important tool, in the context of particle physics. It was introduced in [85] and has already proved itself both highly powerful and useful in

the specialised and highly supersymmetric arena of $\mathcal{N} = 4$ SYM. However, we might anticipate that it will be useful more generally in particle physics (see e.g. [41]).

The symbol associates to any (generalised) polylogarithm, a tensor whose entries are rational functions of the arguments. The rank of the tensor is equal to the weight of the polylogarithm. For example $\log(x)$ is a function of transcendentality one (weight one) and so gives rise to a 1-tensor

$$\mathcal{S}(\log(x)) = x \qquad (2.6.1)$$

If we now consider the classical polylogarithms of transcendentality w, they have a symbol given as

$$\mathcal{S}(\mathrm{Li}_w(x)) = -(1-x) \otimes \overbrace{x \otimes \cdots \otimes x}^{w-1} \qquad (2.6.2)$$

The symbol inherits several properties from logarithms

$$\cdots \otimes xy \otimes \cdots = \cdots \otimes x \otimes \cdots + \cdots \otimes y \otimes \ldots$$
$$\cdots \otimes \frac{1}{x} \otimes \cdots = - \cdots \otimes x \otimes \ldots \qquad (2.6.3)$$

from which naturally follows the key property that the symbol vanishes if and only if any of its entries equal unity

$$\cdots \otimes 1 \otimes \cdots = 0 \qquad (2.6.4)$$

It is also blind to multiplication by constants.

The final property of the symbol we require is to understand how to take the symbol of products of functions. To do this we take the shuffle product of the symbol of each function

$$\mathcal{S}(fg) = \mathcal{S}(f) \amalg \mathcal{S}(g) \qquad (2.6.5)$$

For example,

$$\mathcal{S}(\mathrm{Li}_2(x)\log(y)) = (-(1-x) \otimes x) \amalg y$$
$$= -(1-x) \otimes x \otimes y - (1-x) \otimes y \otimes x - y \otimes (1-x) \otimes x$$
$$(2.6.6)$$

and for three log functions, we have

$$
\begin{aligned}
\mathcal{S}(\log(x)\log(y)\log(z)) &= x \amalg y \amalg z = (x \otimes y + y \otimes x) \amalg z \\
&= x \otimes y \otimes z + x \otimes z \otimes y + z \otimes x \otimes y + y \otimes x \otimes z + y \otimes z \otimes x + z \otimes y \otimes x
\end{aligned}
$$
(2.6.7)

The symbol can be defined recursively and for many purposes it can be useful to do so, for example multiple polylogarithms are motivated more naturally in this manner. One can write the total derivative of any weight-w generalized polylogarithm (here we mean any function with a well-defined rank-w symbol) as follows:

$$
df = \sum_i g_i d\log(x_i)
$$
(2.6.8)

where the g_i are weight $(w-1)$-polylogarithms. Then the corresponding symbol is given as

$$
\mathcal{S}(f) = \sum_i \mathcal{S}(g_i) \otimes x_i
$$
(2.6.9)

This definition together with (2.6.1) gives all the above properties.

Before continuing we should note that the symbol is only sensitive to the highest weight transcendentality part. There is a generalization of these ideas (using Bloch groups) which gives the complete Hopf algebra [81] and as such can determine the lower transcendentality terms too [65]. However in $\mathcal{N} = 4$ all amplitudes at a given loop order are expressed in functions of uniform transcendentality and as such we will not have recourse to explore these ideas here.

The symbol is exceptionally useful since it trivializes very complicated identities involving polylogarithms, reducing them to a linear algebra problem. The most spectacular example of such a simplification, as mentioned earlier, is in reducing the 17 page formula which was computed for the hexagon two-loop Wilson loop in [63, 64] to the single line formula in [85]. However the inverse process whereby one finds the function from the symbol is far from straightforward to do in practice. Indeed the symbol is generally much more complicated and longer than the actual functions which produce it due to the shuffle product and additionally the symbol is also usually non-unique. The great advantage of the special kinematics we consider here is that the functions that occur will turn out to be relatively simple and after obtaining the symbol we are able to reconstruct the functional form for the amplitudes we consider.

There is an additional constraint acting on the symbol whose effect is to ensure that the symbol corresponds to a genuine function, the so-called integrability constraint. The fact that $d^2 f = 0$ together with its recursive definition give these non-trivial and powerful constraints on the symbols of functions. Namely for a weight w-tensor we obtain $(w-1)$ equations

$$\mathcal{S}(f) = \sum x_1 \otimes \cdots \otimes x_w \Rightarrow (x_i \wedge x_{i+1}) \sum_i x_1 \otimes \ldots \hat{x}_i \otimes \hat{x}_{i+1} \otimes \cdots \otimes x_w = 0$$
(2.6.10)

where the hatted terms are omitted from the symbol. We make extensive use of this constraint in deriving our later results. Indeed we next turn to the n-point, 2-loop, MHV calculation as an example using what we have set out thus far, as calculated in [91].

There is still much more currently known as well as to be learnt about the symbol, however to avoid an excess of non-essential background material we will avoid giving additional information and direct the interested reader to the aforementioned references.

2.7   Two-dimensional Kinematics

In this section we give details and conventions for the special kinematics, first introduced in [5], where the external momenta p_i lie entirely in $(1+1)$-dimensions. However first we wish to briefly introduce the concept of a Wilson loop

A Wilson Loop variable is defined as the trace of a path-ordered exponential of a gauge field transported along a closed line C.

$$W_C := \text{Tr} \left(\mathcal{P} \exp \left[i \oint_C A_\mu dx^\mu \right] \right)$$
(2.7.1)

where this quantity has a duality to certain amplitudes calculated in planar $\mathcal{N} = 4$. However, here we are interested in the contour C which has a very simple relationship to the dual-momenta defined in (2.2.1), where it is easy to see that due to conservation of momentum, these variables $x_i^{\alpha\dot{\alpha}}$ provide a polygon with the edges being the momenta. Part of such a contour is shown in Fig.2.1 where in this instance all the momenta lie in $(1+1)$-dimensions and as such we have drawn the figure in light-cone coordinates. The region momenta $x_i^{\alpha\dot{\alpha}}$ (vertices of the corresponding Wilson loop contour) have the following

Figure 2.1: Figure illustrating part of a zig-zag Wilson loop contour in 2d kinematics. Vertices $x_i^{\alpha\dot\alpha}$ are here defined in terms of light-cone coordiantes. In 2d the contour can also be specified by giving every other vertex $x_2, x_4, x_6,...$ and a prescription such as (2.7.2) to differentiate between 'flipped' contours.

form in lightcone coordinates (x_+, x_-):

$$x_i = \begin{cases} (z_{i-1}, z_i)\ , & i \text{ even} \\ (z_i, z_{i-1})\ , & i \text{ odd} \end{cases} \tag{2.7.2}$$

Only an even number of vertices is possible in this 2d kinematics, and we continue denoting it as n (rather than $2n$ as sometimes done in the literature). In our notation z_i components with odd values of i lie along the x^+-axis, and the "even z_i's" are along the x^--axis, as one can see instantly from Fig. 2.1. We will frequently refer to them as 'odd' and 'even' coordinates. All the (bosonic) functions we consider can be written in terms of Lorentz invariant intervals $z_{ij} := z_i - z_j$ where both i, j are either even or odd. In this notation we must remember that the even and odd coordinates are independent of each other.

It is instructive to view the 2-dimensional kinematics from the point of view of momentum twistors (2.2.2). In 2d the bosonic twistors $Z_i^a = (\lambda_i^\alpha, x_{\dot\alpha\alpha,i}\lambda_i^\alpha)$ reduce as follows, for all even values of i we have

$$p_i^{\alpha\dot\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & p_i^- \end{pmatrix} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \quad \Rightarrow \quad \lambda_i^\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ ,\ \tilde\lambda_i^{\dot\alpha} = \begin{pmatrix} 0 \\ p_i^- \end{pmatrix}, \tag{2.7.3}$$

and

$$x_{\dot\alpha\alpha,i}\lambda_i^\alpha = \begin{pmatrix} x_i^+ & 0 \\ 0 & x_i^- \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ z_i \end{pmatrix}. \tag{2.7.4}$$

For odd values of i the story is similar, and as a result, momentum twistors in 2d have a checkered pattern:

$$Z_i = \begin{cases} (Z_i^1, 0, Z_i^3, 0) \equiv (1, 0, z_i, 0) & i \text{ odd} \\ (0, Z_i^2, 0, Z_i^4) \equiv (0, 1, 0, z_i) & i \text{ even ,} \end{cases} \qquad (2.7.5)$$

which is a manifestation of $SU(2,2) \to SL(2)_+ \times SL(2)_-$ in 2d.

In 2d kinematics it is then natural to define an $SL(2)_\pm$-invariant two-bracket of twistors,

$$< ij > := \begin{cases} Z_i^3 Z_j^1 - Z_i^1 Z_j^3 & i \text{ and } j \text{ odd} \\ Z_i^4 Z_j^2 - Z_i^2 Z_j^4 & i \text{ and } j \text{ even} \\ 0 & \text{otherwise} \end{cases} \qquad (2.7.6)$$

From (2.7.6) and the right-hand side of (2.7.5) we have that $< ij > = z_{ij}$ and the Lorentz-invariant intervals z_{ij} have the standard two-bracket interpretation (but in terms of reduced 2d twistors rather than helicity spinors).

Furthermore, the standard $SL(2,2)$-invariant twistor 4-bracket contraction,

$$< ijkl > := \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d , \qquad (2.7.7)$$

reduces in 2d to a product of two-brackets if there are two even and two odd indices, or vanishes otherwise: e.g. $< 1234 > = < 13 > < 24 >$. The principal point here is that lightcone coordinates are interchangeable with twistors in 2d and only two-brackets of bosonic twistors (of the same parity) can appear.

For super-amplitudes in 2d, being "super" i.e. supersymmetric it may be considered natural to consider a supersymmetric reduction, $SU(2,2|4) \to SL(2|2)_+ \times SL(2|2)_-$, under which momentum supertwistors (2.2.2) become [46]

$$\mathcal{Z}_i = (Z_i^a ; \chi_i^A) = \begin{cases} (Z_i^1, 0, Z_i^3, 0; \chi_i^1, 0, \chi_i^3, 0) & i \text{ odd} \\ (0, Z_i^2, 0, Z_i^4; 0, \chi_i^2, 0, \chi_i^4) & i \text{ even ,} \end{cases} \qquad (2.7.8)$$

and we will indeed mostly consider this additional reduction in fermionic coordinates also. On the other hand one should beware that while we may still compute meaningful forms for either $R_{n,k}$ or $\mathcal{R}_{n,k}$ the MHV-prefactor to this from (2.1.6) contains

$$\delta^{(8)} \left(\sum_{i=1}^{n} \lambda_i \eta_i \right) \qquad (2.7.9)$$

which under this SU(4) splitting necessarily goes to zero. For $R_{n,k}$ with $k = 0, 1$ this reduction in the fermionic superspace co-ordinates has no effect beyond a trivial alteration between the way in which the fermionic variables can be contracted, which does not change any of the amplitudes in this case. All results obtained can straightforwardly be uplifted to the case with full fermionic dependence[1]. Beyond NMHV this restriction however does mean a loss of information. Why this matching should be the case at NMHV level is not well understood, but comes from the observed fact that the negative helicity must, for some unknown reason, be split equally between even-labelled and odd-labelled particles.

The remainder function $\mathcal{R}_n$ is dual conformally invariant [59, 61] and as such its lowest bosonic component ($\mathcal{R}_{n,0}$) can be written as a function of cross-ratios. The non-MHV components ($\mathcal{R}_{n,k>0}$) also depend on superconformal invariants involving Grassmann variables. We first concentrate on the purely bosonic case of the MHV amplitudes.

We define the most general cross-ratios in special 2d kinematics as

$$u_{ij;kl} = \frac{<il><jk>}{<ik><jl>} \, . \tag{2.7.10}$$

This equation is meaningful only for i, j, k, l being ALL odd or ALL even. In other words, all the cross-ratios fall into two separate classes with all indices being even or with all indices odd. Cross-ratios with indices of mixed parity (even and odd) do not exist.

The general cross-ratios in 2d kinematics also satisfy an additional algebraic constraint,

$$u_{i,j;k,l} = 1 - u_{i,l;k,j} \, . \tag{2.7.11}$$

Fundamental cross-ratios are given by,

$$u_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2} = \frac{\langle i-1, j+1\rangle \langle i+1, j-1\rangle}{\langle i-1, j-1\rangle \langle i+1, j+1\rangle} = u_{i-1,i+1;j-1,j+1} \, . \tag{2.7.12}$$

For the lowest in n cases, $n = 8$ and $n = 10$, all non-trivial 2-component cross-ratios are of the form $u_{i,i+4}$, with $i = 1, \ldots, 4$ for the octagon, and $i = 1, \ldots, 10$ for the decagon with the additional constraint:

$$n = 8 \quad : \quad 1 - u_{i,i+4} = u_{i+2,i+6} \, , \qquad\qquad i = 1, 2 \tag{2.7.13}$$

$$n = 10 \quad : \quad 1 - u_{i,i+4} = u_{i+2,i+6} \, u_{i-2,i+2} \, , \qquad i = 1, \ldots, 10. \tag{2.7.14}$$

---

[1] I.e. non-vanishing $(\chi^1, \chi^2, \chi^3, \chi^4)$ in both equations (2.7.8)

At $n = 8$ points there are just four fundamental cross-ratios, u_1, u_2, v_1 and v_2:

$$u_1 := u_{1,5} \quad , \quad u_2 := u_{2,6} \quad , \quad u_3 := 1 - u_1 := v_1 \quad , \quad u_4 := 1 - u_2 := v_2 \, .$$
(2.7.15)

In order to relate general cross-ratios to this reduced set we may use

$$u_{i,j;k,l} = \prod_{I=i+1}^{j-1} \prod_{K=k+1}^{l-1} u_{I,K} \, .$$
(2.7.16)

This reduced set of $u_{i,j}$ cross-ratios also has a 4d interpretation (2.7.12), $u_{i,j} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2}$.

For the two lowest-n cases (the octagon and the decagon), all non-trivial 2-component cross-ratios are of the form $u_{i,i+4}$ (2.7.13), with $i = 1, \ldots, 4$ for the octagon and $i = 1, \ldots, 10$ for the decagon. The cross-ratios $u_{i,j}$ are still not all independent because of equations (2.7.11), leaving $n - 6$ (i.e. 2 for the octagon and 4 for the decagon) independent solutions. For the octagon (2.7.11) amounts to

$$n = 8 \ : \quad 1 - u_{i,i+4} = u_{i+2,i+6} \, , \quad i = 1, 2 \, ,$$
(2.7.17)

and for the decagon,

$$n = 10 \ : \quad 1 - u_{i,i+4} = u_{i+2,i+6} \, u_{i-2,i+2} \, .$$
(2.7.18)

To simplify notation at low n, it is sometimes convenient to use

$$u_i := u_{i,i+4} \, .$$
(2.7.19)

While at $n = 8$ and $n = 10$ these are the only cross-ratios, this is no longer true at $n \geq 12$ where $u_{i,j}$ cross-ratios appear with $j - i \geq 6$. More details on the cross-ratios in the special kinematics can be found in [89].

2.8   COLLINEAR LIMITS IN THE 2D KINEMATICS

From the zig-zag kinematics it is clear that the lowest collinear limit one can apply and remain within the $(1+1)$-dimensional kinematics is the triple collinear limit, where three consecutive edges collapse into one. The reason for this is that amplitudes in two-dimensional kinematics require an even number of external particles and as such a simple double-collinear limit would require us
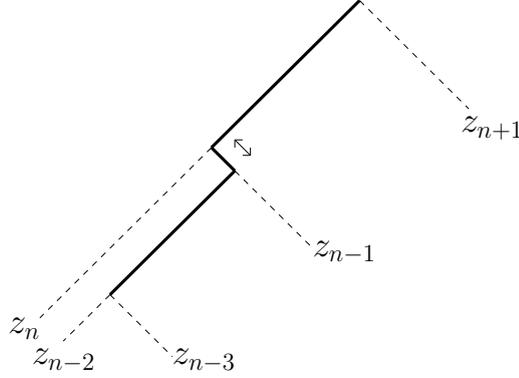
Figure 2.2: Figure illustrating the triple/soft collinear limit $z_n \to z_{n-2}$ while the variable z_{n-1} remains free.

taking two momenta restricted to opposite lightcone directions and taking the limit as they approach each other, which would necessarily be as they both approached 0. This would not be a collinear limit, in fact it would be a double-soft limit and as such we will only consider collinear limits of an odd-number of particles. More precisely a triple collinear limit this should be thought of as a collinear-soft-collinear limit, where three edges with momenta p_{n-2}, p_{n-1} and p_n collapse into a single edge p_{n-2}. In practice, in a triple collinear limit the middle momentum becomes soft, $p_{n-1} \to 0$. In terms of twistors, or the light-cone components z_i's, we see that $z_n \to z_{n-2}$ while the variable z_{n-1} remains free, as demonstrated in Fig 2.2.

We will ultimately be writing our amplitudes in terms of cross-ratios $u_{i,j}$ and as such it is important to understand how the triple collinear limits in particular act on these objects. Naturally any collinear limit which leaves all terms $i-1, i+1, j-1$ and $j+1$ unaffected will have no effect on the cross-ratio. However, if we have any collinear limit which acts across i or j, that is $i-1 \leftrightarrow i+1$ and $j-1 \leftrightarrow j+1$ send:

$$\lim_{i-1 \to i+1} u_{i,j} = \lim_{i-1 \to i+1} \frac{\langle i-1, j+1 \rangle \langle i+1, j-1 \rangle}{\langle i-1, j-1 \rangle \langle i+1 j+1 \rangle} = \frac{\langle i+1, j+1 \rangle \langle i+1 j-1 \rangle}{\langle i+1 j-1 \rangle \langle i+1 j+1 \rangle} = 1$$
$$(2.8.1)$$

If we next consider a collinear limit which takes $i+1 \leftrightarrow i+3$ then we have a simple modification for the cross-ratio which generically will be well-defined and non-zero, with the exception being any collinear limit sending $i+1 \leftrightarrow j-1$ or $j+1 \leftrightarrow i-1$, where the term would go to 0.

The final collinear limit will be where e.g. $i-1$ becomes soft such as when $i-2 \leftrightarrow i$, this in turn sends $i-1 \to 0$ (this is the soft-limit implicit in our triple collinear limit) which could potentially lead to an ill-defined term $\log(u_{i,j}) =$

$\log\left(\frac{\langle i-1,j+1\rangle\langle i+1,j-1\rangle}{\langle i-1 j-1\rangle\langle i+1,j+1\rangle}\right)$. As such, we require any term with $i-1$ in the argument of the symbol entry to drop out, that is we require multiplication by another term which goes to zero under $i-2 \leftrightarrow i$ etc. We use these ideas explicitly when we derive results in two-dimensional kinematics both below and in later chapters and hope that through later use the brief explanation here gains additional clarity.

In the 2d kinematics there are no non-trivial cross-ratios at 6-points (lowest non-trivial case being $\mathcal{R}_{8,0}$). Additionally the 6-point remainder amplitude $\mathcal{R}_6$ is a (coupling dependent) constant multiplied by the tree-level amplitude, which can be reabsorbed into $\mathcal{R}_n$ which we now call $\tilde{\mathcal{R}}_n$ [91].

$$\tilde{\mathcal{R}}_n = \mathcal{R}_n - \frac{n-4}{2}\,\mathcal{R}_6\,, \tag{2.8.2}$$

So at the level of super-amplitudes and for all triple collinear limits, we have

$$\tilde{\mathcal{R}}_n \rightarrow \tilde{\mathcal{R}}_{n-2}\,. \tag{2.8.3}$$

We remark here that for $\mathcal{R}_n = \log R_n$ expressions at different order in the loop expansion do not mix,

$$\tilde{\mathcal{R}}_n^{(\ell)} \quad \rightarrow \quad \tilde{\mathcal{R}}_{n-2}^{(\ell)}\,, \tag{2.8.4}$$

$$\tilde{\mathcal{R}}_n^{(\ell)} \quad \rightarrow \quad \tilde{\mathcal{R}}_{n-m}^{(\ell)} + \tilde{\mathcal{R}}_{m+4}^{(\ell)}\,, \quad \text{for } m \geq 4\,, \tag{2.8.5}$$

and thus $\tilde{\mathcal{R}}_n$ is the natural object to use for collinear uplifts of amplitudes to higher number of points. Before we continue, let us first clarify what precisely we mean by the term, "uplift". By uplift we intend that we wish to write down the n-point amplitude in such a way that the reduction under a triple collinear limit to the $(n-2)$-point amplitude is manifest. So if we find the correct combination of the $(n-2)$-, $(n-4)$-,... point pieces and an additional vanishing part, then we can construct the n-point amplitude in this way, where collinear limit restrictions are manifest. For example the 10-point, 2-loop MHV amplitude in 2d kinematics must be a combination of the 6- and 8-point pieces and a part which vanishes in all collinear limits as we shall see next.

2.9 2-loop, n-point MHV Amplitude in Reduced Kinematics

Before we turn our attentions to the two-loop amplitude, we should first ask ourselves why not start with the one-loop case? We first demonstrate that there can be no non-trivial kinematic function at 6-points in 2d kinematics and as such the more detailed analysis can commence at 8-points. We show that such a function (i.e. an 8-point, 1-loop function) cannot exist, since under the simple triple collinear limit we should obtain the 6-point, 1-loop MHV amplitude. However this must necessarily be constant since all cross-ratios at 6-points go to a constant in the 2d kinematical limit:

$$\frac{\langle 1,2,3,4\rangle\langle 4,5,6,1\rangle}{\langle 1,2,4,5\rangle\langle 3,4,6,1\rangle} \rightarrow \frac{\langle 1,3\rangle\langle 5,1\rangle\langle 2,4\rangle\langle 4,6\rangle}{\langle 1,5\rangle\langle 3,1\rangle\langle 2,4\rangle\langle 4,6\rangle]} = 1 \tag{2.9.1}$$

etc. As such we can have no conformal cross-ratios at 6-points in our reduced kinematics, meaning all remainder functions at 6-points must be a constant with respect to kinematics.

Before continuing let us briefly set out the basis of two-dimensional cross-ratios at eight-points and discuss their behaviour under our standard triple collinear limits. At eight-points there are four cross-ratios all of the form $u_{i,i+4}$:

$$u_{1,5} \qquad u_{3,7} \qquad u_{2,6} \qquad u_{4,8} \tag{2.9.2}$$

These cross-ratios are related to one another by the simple equations

$$u_{1,5} = 1 - u_{3,7} \qquad u_{2,6} = 1 - u_{4,8} \tag{2.9.3}$$

as such when $u_{1,5} \rightarrow 0$ under a triple collinear limit then necessarily $u_{3,7} \rightarrow 1$ and vice versa. As such if we have $u_{1,5}$ in the symbol then we necessarily need $u_{3,7}$ in the symbol such that the term vanishes whenever either $u_{1,5}$ or $u_{3,7}$ goes to 0, and naturally there is an analogous statement about $u_{2,6}$ and $u_{4,8}$. So let us briefly see what happens to $u_{1,5}$ under all 8 possible triple collinear limits since the other results can be deduced from this simply by cyclicity. Let us remind ourselves that we defined $u_{1,5}$ as:

$$u_{1,5} = \frac{\langle 8,6\rangle\langle 2,4\rangle}{\langle 8,4\rangle\langle 2,6\rangle} \tag{2.9.4}$$

and this remains completely unaffected by all limits acting solely on odd-labelled particles e.g. $\lim_{1\to3}$. So let us write down what happens to this term under the other four triple collinear limits:

$$\lim_{2\leftrightarrow4} u_{1,5} = 0 \qquad \lim_{4\leftrightarrow6} u_{1,5} = 1 \qquad \lim_{6\leftrightarrow8} u_{1,5} = 0 \qquad \lim_{8\leftrightarrow2} u_{1,5} = 1 \qquad (2.9.5)$$

So as we can see $u_{1,5}$ goes to 1 under any triple collinear limit across 1 or 5 but goes to zero in triple collinear limits across 3 or 7. In general then our result which we shall use from this point forward is simply:

$$\lim_{i-1\leftrightarrow i+1} u_{i,j} = 1 = \lim_{j-1\leftrightarrow j+1} u_{i,j} \qquad (2.9.6)$$

Now at 8-points one-loop, it is very simple to see that there is no weight-two symbol (meaning no 2-tensor) which we can write down and which vanishes in all triple collinear limits. For example if we wrote $u_{3,7} \otimes u_{1,5}$ then by our definition (2.9.6) this would vanish in all triple collinear limits across 1,3,5 and 7 e.g. $8 \to 2$ and $4 \to 6$ etc. However it is only a function of even-points and as such is left untouched by collinear limits such as $1 \to 3$. This very easily demonstrates that in these reduced kinematics there can be no one-loop remainder function at 8-points and by a simple induction argument, for any number of external points.

We approach the question of the 8-point 2-loop remainder function in the same vein, we now try to write a weight-4 symbol which vanishes in every collinear limit. Note that here we want to write down the most general weight-4 symbol we can, but it must obey the dihedral symmetry (cyclicity and parity) in addition to vanishing in all collinear limits and then also being 'integrable'. We will assume, as was done in the derivation [89, 91], that all entries of the symbol are cross-ratios. This assumption has later been shown to be too restrictive for all two-dimensional kinematic amplitudes [47] and we need to allow certain linear combinations of cross-ratios. Note that these linear-combinations do not allow any single entry to vanish under more than two collinear limits, to be explicit this means a symbol of weight n can only ever disappear in a maximum of $2n$ distinct triple collinear limits. As such we will first begin using the original assumptions of [89, 91] and then discuss possible additional terms once we relax this assumption.

It is simple to see that if entries can only contain cross-ratios then every symbol must contain all four of the cross-ratios at 8-points. If any $u_{i,i+4}$ cross-ratio is missing then the triple collinear limit across i and $i + 4$ will not go

to a constant (2.9.6). For example if we take $u_{2,6} \otimes u_{2,6} \otimes u_{4,8} \otimes u_{1,5}$ then, as discussed above, under the collinear limit $2 \to 4 \Rightarrow u_{1,5} \to 0$, but we do not have the cross-ratio $u_{3,7} \to 1$ and this term 'blows up'. Therefore we clearly require inclusion of all cross-ratios and so we consider the 4! different symbols which contain all 4 of the cross-ratios. Imposing the dihedral symmetry we obtain only 3 non-trivial, independent terms and a constant $\mathcal{R}_6^{(2)}$:

$$\mathcal{R}_8^{(2)} = a\mathcal{X}_8^{(2)} + b\mathcal{Y}_8^{(2)} + c\mathcal{Z}_{8)}^{(2)} + 2\mathcal{R}_6^{(2)} \tag{2.9.7}$$

where

$$\mathcal{S}(\mathcal{X}_8^{(2)}) = u_{1,5} \otimes u_{2,6} \otimes u_{3,7} \otimes u_{4,8} + 7 \text{ terms related by dihedral symmetry}$$
$$\mathcal{S}(\mathcal{Y}_8^{(2)}) = u_{1,5} \otimes u_{2,6} \otimes u_{4,8} \otimes u_{3,7} + 7 \text{ terms related by dihedral symmetry}$$
$$\mathcal{S}(\mathcal{Z}_8^{(2)}) = u_{1;5} \otimes u_{3,7} \otimes u_{2,6} \otimes u_{4;8} + 7 \text{ terms related by dihedral symmetry}$$

$$\tag{2.9.8}$$

All three terms separately vanish in the collinear limit, and are symmetric under the full dihedral symmetry.

We now need to consider whether these symbols match to any actual function, that is, do they satisfy the integrability condition? We will derive the first of these integrability conditions carefully and leave the other two as an exercise. Let us consider the integrability condition (2.6.10) based on the derivatives hitting the first two entries. When the derivatives act on the term $S\left(\mathcal{X}_8^{(2)}\right)$ we gain the term:

$$a\frac{du_{1,5} \wedge du_{2,6}}{u_{1,5}u_{2,6}}u_{3,7} \otimes u_{4,8} \tag{2.9.9}$$

since $d\log\left(u_{1,5}\right) = \frac{du_{1,5}}{u_{1,5}}$ etc. We follow the same process when the derivatives act on the first two entries of the other two terms and this gives us our first restriction:

$$a\frac{du_{1,5} \wedge du_{2,6}}{u_{1,5}u_{2,6}}u_{3,7} \otimes u_{4,8} + b\ \frac{du_{1,5} \wedge du_{2,6}}{u_{1,5}u_{2,6}}u_{4,8} \otimes u_{3,7}$$
$$+c\ \frac{du_{1,5} \wedge du_{3,7}}{u_{1,5}u_{3,7}}u_{2,4} \otimes u_{3,7} + \text{dihedral} = 0 \tag{2.9.10}$$

Where "+ dihedral" represents all the additional terms related to these by dihedral symmetry. We obtain two more equations from the derivatives hitting

the first two entries, the second and third entries and the final two entries.
These restrictions are:

$$a\frac{du_{2,6} \wedge du_{3,7}}{u_{2,6}u_{3,7}}u_{1,5} \otimes u_{4,8} + b \ \frac{du_{2,6} \wedge du_{4,8}}{u_{2,6}u_{4,8}}u_{1,5} \otimes u_{3,7}$$

$$+c \ \frac{du_{3,7} \wedge du_{2,6}}{u_{3,7}u_{2,6}}u_{1,5} \otimes u_{4,8} + \text{dihedral} = 0$$

$$a\frac{du_{3,7} \wedge u_{4,8}}{u_{3,7}u_{4,8}}u_{1,5} \otimes u_{2,6} + b \ \frac{du_{4,8} \wedge du_{3,7}}{u_{4,8}u_{3,7}}u_{1,5} \otimes u_{2,6}$$

$$+c \ \frac{du_{2,6} \wedge du_{4,8}}{u_{2,6}u_{4,8}}u_{1,5} \otimes u_{3,7} + \text{dihedral} = 0$$

$$(2.9.11)$$

Now we consider the wedge terms, and since $u_{1,5} = 1 - u_{3,7}$ and $u_{2,6} = 1 - u_{4,8}$ then $du_{1,5} = -du_{3,7}$ and $du_{2,6} = -du_{4,8}$. The minus sign disappears at the level of the symbol as it is blind to constants but this means there is only independent wedge product $du_{1,5} \wedge du_{2,6}$, since:

$$du_1 \wedge du_3 = du_2 \wedge du_4 = 0 \quad du_1 \wedge du_2 = du_1 \wedge du_4 \quad \text{etc.} \quad (2.9.12)$$

So for example (2.9.10) becomes:

$$(a - b)\frac{du_{1,5} \wedge du_{2,6}}{u_{1,5}u_{2,6}}(u_{1,5} \otimes u_{2,6} - u_{2,6} \otimes u_{1,5} + u_{2,6} \otimes u_{3,7} - u_{3,7} \otimes u_{2,6} +$$

$$u_{3,7} \otimes u_{4,8} - u_{4,8} \otimes u_{3,7} + u_{4,8} \otimes u_{1,5} - u_{1,5} \otimes u_{4,8}) \quad (2.9.13)$$

The other restrictions are analogous and fix the result $a = b = c$.

This results in the symbol for $\log(u_{1,5}) \log(u_{2,6}) \log(u_{3,7}) \log(u_{4,8})$. As such it is therefore clear that the suprisingly simple form for the 8-point 2-loop remainder function is fixed to be:

$$\mathcal{R}_8^{(2)} = a(\mathcal{X}_8^{(2)} + \mathcal{Y}_8^{(2)} + \mathcal{Z}_8^{(2)}) + 2\mathcal{R}_6^{(2)}$$

$$= a\log(u_{1,5}) \log(u_{2,6}) \log(u_{3,7}) \log(u_{4,8}) + 2\mathcal{R}_6^{(2)} \quad (2.9.14)$$

which agrees with known results [91] where $a = -\frac{1}{2}$ and $\mathcal{R}_6^{(2)} = -\frac{\pi^4}{36}$. As such we have been able to derive the 8-point 2-loop remainder function purely from symmetry considerations and some assumptions about the entries of the symbol up to two un-fixed coefficients, of which one is merely the 6-point 2-loop constant.

The question now becomes once we relax the assumption as done in [47]

what additional terms could we include in this analysis? Note that the result of $R^{(2)}_{n;0}$ from [89] has been shown to be correct, numerically and with other non-trivial checks, as such if we find that under our symmetry restrictions there are additional permitted terms, this indicates there must be additional restrictions preventing these terms from appearing. For example it is simple enough to see that the term $u_{1,5} \otimes u_{3,7} \otimes \frac{u_{1,5}-u_{2,6}}{u_{3,7}} \otimes u_{4,8}$ satisfies the property that it vanishes in all triple collinear limits (2.9.6), so what is the reason that such a term does not appear in the 8-point MHV remainder function? We could also use entries of this more complicated form $\frac{v-w}{1+v}$ in more than one position, indeed we could have all four entries of the symbol string being of this more complicated form, provided it still vanishes under collinear limits. In [47] an additional assumption was made when deriving the 2-loop NMHV amplitude that this is not the case and these terms appear in only one position. This led to the correct results for the 2-loop NMHV amplitude and in turn the 3-loop MHV amplitude as checked numerically, however there seems to be no proof of this yet. We use this assumption and show therefore that these terms cannot be included in the two-loop MHV remainder function.

There are important restrictions on where we can and cannot place these additional, more complicated terms within the symbol string. Firstly as was explained in [54], where the authors working in four-dimensions built the symbol of the four-point amplitude at three-loops dealing with terms of our new form, we note that these terms cannot occur in the first position. The reason for this is simply that the first entries of a symbol determines the branching points of the function, and the symbol of the discontinuity across the branch cut is obtained by dropping this first entry from the symbol. For Feynman integrals without internal masses the branch cuts extend between points where one of the Mandelstam invariants $= 0, \infty$. As such, the first entries of the symbol must be distances between two points x^2_{ij}, which combined with conformal invariance means that the first-entries of conformally invariant functions must be cross-ratios. Equally as in [45] we note that the branch cuts themselves have branch cuts, the kinematical interpretation of which is more complicated, but dictate the location of the endpoints can involve at most one extra channel and therefore should be similar to the first entry, i.e. the second entry of a symbol of a conformal function can only be a cross-ratio. It is not until the third position where the branch cut discontinuity may depend on up to three channels that we can have more complicated entries, these ideas being utilised in [78] as well as elsewhere.

Additionally, following [53] we will assume that the last entry of the symbol has restrictions preventing the placing of these more complicated terms there also. The observations of [45] suggests that this fact may be related to a supersymmetric formulation of the Wilson loop. These observations suggest that the full symbol of any conformal function should also have its final symbol entry drawn from the cross-ratios alone. As such we now know that these more complicated entries cannot appear in the first, second or last slots of our symbol, meaning at two-loops they can only appear in the third position. Now it is simple to see that they disappear under the integrability condition. To see this consider the single term

$$\alpha \left[u_{4,8} \otimes u_{3,7} \otimes (u_{1,5} - u_{2,6}) \otimes u_{1,5} \right] \tag{2.9.15}$$

and take the integrability condition on the final two entries gives us a contribution

$$\alpha \frac{-du_{2,6} \wedge du_{1,5}}{(u_{1,5} - u_{2,6}) u_{1,5}} u_{4,8} \otimes u_{3,7} \tag{2.9.16}$$

for this term to be cancelled we would need another term to give the term with the opposite sign. This would clearly require another symbol entry $u_{1,5} - u_{2,6}$ which can only go in the third spot and then the rest is fixed to be identical to the term above. As such the only solution is for $\alpha = 0$, and these terms not to contribute here, however for the three-loop MHV amplitude these terms do indeed contribute.

We will consider the question of building the amplitude at 10-points and beyond in two stages and this approach will be a blueprint of sorts for later work. We first ask what functions we could add at 10-points such that they disappear in all possible collinear limits. Secondly we ask what is the correct way to "uplift" the 8-point amplitude so that in any triple collinear limit we recover the correct 8-point amplitude. The generalisation of this process to higher-loop order, higher helicity configurations and higher points is the principal result of the first half of this thesis. However the result for purely MHV amplitudes was written down in [89] and [91] first, before it was generalised in [80] and as such we will give the result for this case before moving on to the question of implementing the 2-dimensional restrictions to NMHV amplitudes in the next chapter.

There is no weight-4 tensor which disappears in all triple collinear limits at ten-points. If we apply a similar analysis to that we performed at eight-points we can see that we would need all five cross-ratios inside the symbol (2.9.6),

as such we would require a symbol string of weight 5. For example the term $u_{1,5} \otimes u_{3,7} \otimes u_{2,6} \otimes u_{4,8} := S$ where these indices are understood mod(10), this term does not vanish in the triple collinear limits $8 \leftrightarrow 10$ and $9 \leftrightarrow 1$:

$$\lim_{10 \to 8} S = \frac{\langle 8,6 \rangle \langle 2,4 \rangle}{\langle 8,4 \rangle \langle 2,6 \rangle} \otimes u_{3,7} \otimes u_{2,6} \otimes u_{4,8} \neq 0 \qquad (2.9.17)$$

Once again we note that relaxing the condition on the entries of the symbol to contain particular linear-combinations of cross-ratios cannot contribute to terms vanishing under additional collinear limits. This means that a weight four symbol can never disappear under more than 8 distinct triple collinear limits. It is clear that there will be no collinear-vanishing term at ten-points or indeed at any higher number of points. This means the entire problem of solving the n-point, 2-loop, MHV remainder function is merely the problem of correctly uplifting the 8-point answer. It is known from [91] and [89] that the correct way to do this is simply:

$$\mathcal{R}_{10}^{(2)} = -\frac{1}{2}(\log(u_{10,2;4,6})\log(u_{1,3;5,7})\log(u_{2,4;6,10})\log(u_{3,5;7,1}) + \text{cyclic}) - \frac{\pi^4}{12} \tag{2.9.18}$$

where we recall this definition of the cross-ratios from (2.7.12). Note that the term written out explicitly, under the triple collinear limit $10 \to 8$ goes to:

$$\mathcal{R}_8^{(2)}(1,2,3,4,5,6,7,8) = -\frac{1}{2}(\log(u_{8,2;4,6})\log(u_{1,3;5,7})\log(u_{2,4;6,8})\log(u_{3,5;7,1})) - \frac{\pi^4}{12} \tag{2.9.19}$$

It is simple to check that all other terms either go to zero or cancel with one another in this limit and a similar result can trivially be concluded for all other triple collinear limits, due to cyclicity.

As such we conclude that the 2-loop, 10-point remainder function is uniquely fixed by the 8-point function as found in [89, 91], which satisfies the correct collinear limits and symmetries. This same analysis can be performed for all higher points and as such we can obtain the n-point 2-loop remainder function in the following way, if we label:

$$S_8^{(2)}(x_i, x_j, x_k, x_l) = \mathcal{R}_8^{(2)}(i-1, i, j-1, j, k-1, k, l-1, l) \qquad (2.9.20)$$

then the n-point answer can be written

$$\mathcal{R}_n^{(2)} = \sum_{i \lhd j \lhd k \lhd l \lhd i} S_8^{(2)}(x_i, x_j, x_k, x_l) + \text{constant} \qquad (2.9.21)$$

where the notation $i \triangleleft j$ means $i+1 < j$, and the sum is understood to be cyclic. This is the first and simplest manifestation of the general uplifting formula which we will now dedicate the next two chapters to deriving as in [80]. We first show how to include NMHV amplitudes in this analysis. Then, revisiting this example and defining the notation more generally, we will give a general formula for determining all n-point, N^kMHV, ℓ-loop amplitudes up to unfixed constants, as an uplift of lower-point amplitudes and collinear vanishing parts and discuss the recipe for determining these pieces.

3

NMHV One-Loop Amplitudes in Two-Dimensional Kinematics

3.1 NMHV Amplitudes and R-Invariants

Let us start by considering the principal difference between the MHV and the NMHV remainder functions, namely the introduction of additional fermionic variables which count the extra negative helicity state. All of these amplitudes are of order η^{12} (in the Taylor expansion in Grassmann variables) and as such, once we move to the remainder function and remove the MHV prefactor and with it a polynomial of η^8, we are left with terms proportional to η^4. We know from our earlier discusssion of the superamplitude Sect 2.1 that all these amplitudes are supersymmetrically related to the 3-negative gluon amplitude, which will be proportional to η_i^4, where i labels the additional negative helicity gluon. We also are aware that the amplitude must be dual-conformally invariant. From this the question becomes how do we write down dual conformal invariant functions which supersymmetrically contain all the additional helicity information for these NMHV amplitudes?

These invariant functions are known as "R-invariants" first derived in [59] and then recast in momentum-supertwistor space in [100] and [8] which develops a geometrical picture of the R-invariants as polytopes which we will make use of and which was a fore-runner of the recent Amplituhedron. We avoid a long derivation, referring the interested reader to the above references, instead we will simply give a definition and exploration of the relevant features.

An R-invariant is a dual-conformally invariant function of both bosonic and fermionic variables as given in [100]. Note that four-brackets are functions of

momentum twistors, and we use short-hand e.g. $1 \equiv \mathcal{Z}_1$ etc.

$$R_{1,2,3,4,5} = R(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5)$$
$$:= \frac{\delta^{0|4}(\chi_1\langle 2,3,4,5\rangle + \text{cyclic})}{\langle 1,2,3,4\rangle\langle 2,3,4,5\rangle\langle 3,4,5,1\rangle\langle 4,5,1,2\rangle\langle 5,1,2,3\rangle} \qquad (3.1.1)$$

where most of this notation should be familiar, however the numerator contains a fermionic δ-function of χ-(Grassmann-)variables which we shall explain. A fermionic δ-function acts in much the same way as a normal δ-function, in the equation above it imposes the constraint $(\chi_1\langle 2,3,4,5\rangle + \text{cyclic})^4 = 0$, which enforces all the supersymmetric relations between different helicity states in the superamplitude. The χ-variables are related to our earlier fermionic variables η through the definitions (2.2.1),(2.2.2) which we reproduce:

$$\lambda_i^\alpha \eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} \qquad A = 1,2,3,4$$
$$\mathcal{Z}_i = (Z_i^\alpha, \chi_i^A) = (\lambda_i^\alpha, x_{\alpha\dot\alpha,i}\lambda_i^\alpha; \theta_{\alpha,i}^A\lambda_i^\alpha) \qquad (3.1.2)$$

Returning to (3.1.1) we note that this function has five poles, these being whenever any of the 4-brackets in the denominator go to zero. Some of these poles are physical poles of the amplitude i.e all those of the form $\langle i, i+1, j, j+1\rangle \to 0$, corresponding to $p_i \cdot p_j \to 0$. Those not of this form are spurious poles which should drop out in the complete amplitude and we will revisit these in the next couple of subsections. Suffice to say that such spurious poles will cause problems term-by-term when we restrict our data to the two-dimensional kinematics where (for example at 6-points) $\langle 5,1,2,3\rangle \to 0$.

Let's now consider the way in which we will use these R-invariants to construct amplitudes, the simplest way to see this is to start by calculating tree-level amplitudes. We start with the 5-point NMHV amplitude which is simply $R_{1,2,3,4,5} \times$ MHV. It is non-trivial to check that this is just the parity-flip of the MHV amplitude which it must be at 5-points, since NMHV $\equiv \overline{\text{MHV}}$. At 6-points we have many ways of expressing the tree-level NMHV remainder function due to non-trivial linear identities which the R-invariants obey, however the two we will concentrate on are:

$$\begin{aligned} R_{6,1}^{(0)} &= R_{1,2,3,4,5} + R_{3,4,5,6,1} + R_{5,6,1,2,3} \\ &= R_{2,3,4,5,6} + R_{4,5,6,1,2} + R_{6,1,2,3,4} \end{aligned} \qquad (3.1.3)$$

we note here that there should be no confusion in notation between R-invariants

and the notation for the remainder function since the indices distinguish one from the other, for example the superscript (0) denotes that this is a tree-level amplitude. We also will often use $[1, 2, 3, 4, 5] := R_{1,2,3,4,5}$ as defined in (3.1.1) as elsewhere in the literature.

We first encounter non-trivial linear relations at 6-points as we noted above, so if we label the R-invariants at 6-points by their missing argument, we can write the relation (3.1.3) as

$$(1) + (3) + (5) = (2) + (4) + (6) \tag{3.1.4}$$

To check this equality in generality is highly non-trivial and it is not until later in the chapter will we develop a method to check whether any two sets of R-invariants are equivalent or not. Here we simply remark upon it as a curious aside, lacking a unique way to write our amplitude will potentially be an issue in our later analysis. For now though let us move on and see how these R-invariants arise in the loop-level amplitudes.

In [9] we find an expression for the 5- and 6-point, one-loop NMHV amplitude remainder as follows:

$$R_{5,1}^{(1)} = [1, 2, 3, 4, 5] \left(\int_{AB} \frac{\langle AB13\rangle\langle 1245\rangle\langle 2345\rangle - \langle AB(512) \cap (123)\rangle\langle 1345\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle} \right) \tag{3.1.5}$$

where $(512) \cap (123)$ denotes the intersection of these two planes in momentum-twistor space, for calculational purposes we can re-express it simply as $\langle 5121\rangle\mathcal{Z}_2\mathcal{Z}_3 - \langle 5122\rangle\mathcal{Z}_3\mathcal{Z}_1 + \langle 5123\rangle\mathcal{Z}_1\mathcal{Z}_2$ which naturally simplifies as the first two terms are both zero.

$$R_{6,1}^{(1)} = (1+g+g^2)\, [(1) - (2) + (3)] \int_{AB} \frac{\langle AB13\rangle\langle AB46\rangle\langle 5612\rangle\langle 2345\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB56\rangle\langle AB61\rangle} \tag{3.1.6}$$

where $g : i \rightarrow i + 1$ acts on everything to its right, both R-invariants (here labelled as in (3.1.4) by the particle missing from argument) and integral. In the integrals above there must be consideration made for the correct measure and manner in integrating over the pair of twistors AB, however such a discussion would take us too far from our main aims and we will not elaborate on it here, instead leaving it to [9, 100] and other associated papers.

In this chapter we will derive the one-loop analytic NMHV amplitudes in 2d kinematics purely from symmetry constraints. Not only do we avoid any integration, but we will find a very simple uplift to all number of external

points, and we will draw comparison between this and the uplift of the two-loop MHV amplitude, thus motivating the search for a universal uplift in the next chapter. First though we will have to deal with the issue of spurious poles and it is to this we now turn our attention.

3.2 A Finite Basis of R-invariants in 2d

Let us begin this section by considering the NMHV, tree-level amplitudes which can be written in the BCFW form [39]:

$$R_{n,1}^{(0)} = \sum_{1 < i < j \leq n} R_{1,i-1,i,j-1,j} \tag{3.2.1}$$

Where we defined the R-invariants before this in (3.1.1). From this we concentrate first on the 6-point, NMHV, tree-level amplitude which is written in equation (3.1.3). Let us take the first term $R_{1,2,3,4,5}$ and consider what poles it has which are physical, which poles are unphysical but finite and which poles are unphysical and 'blow up' in 2d kinematics. We recall (3.1.1) that:

$$R_{1,2,3,4,5} = \frac{\delta^{0|4}(\chi_1 \langle 2,3,4,5 \rangle + \text{cyclic})}{\langle 1,2,3,4 \rangle \langle 2,3,4,5 \rangle \langle 3,4,5,1 \rangle \langle 4,5,1,2 \rangle \langle 5,1,2,3 \rangle} \tag{3.2.2}$$

At six-points the only physical poles - i.e. ones which correspond to a Feynman graph channels - are of the form $\langle i, i+1, j, j+1 \rangle = \langle 1,2,3,4 \rangle$ and $\langle 2,3,4,5 \rangle$. This leaves us with three 'spurious' poles, which means that they do not correspond to a channel in the Feynman graphs and as such should not be poles of the complete amplitude, i.e. they should drop out of the overall sum. These three poles are split into two groups, recall that four-brackets are only non-zero if there are two-odd and two-even particles (2.7.7). The first group is poles which are spurious but do not 'blow-up' in the two-dimensional kinematics $\langle 4,7,1,2 \rangle$. The second and more interesting group of two-poles are those which are both spurious and blow-up in the two-dimensional kinematics $\langle 3,4,5,1 \rangle$ and $\langle 5,1,2,3 \rangle$, and we now consider these one at a time.

Firstly we ask where else we could obtain a term which has the pole $\langle 3,4,5,1 \rangle$? It should be clear that this can only come from two terms if we require them to be of the form used in (3.2.1), the one we started with $R_{1,2,3,4,5}$ and a new term $R_{1,3,4,5,6}$. As such given that this is a spurious pole there would be a kinematical choice where this pole and only this pole diverges and as such it must be this combination which is free of this spurious pole. Similarly if we

consider our second pole $\langle 5, 1, 2, 3 \rangle$ it is clear from an equivalent argument that this must be cancelled using a term $R_{1,2,3,5,6}$. From this argument it is apparent that when one has a divergent pole there is a unique term in the BCFW expansion (3.2.1) which must be included to cancel that divergent pole. So we now have the combination

$$R_{1,2,3,4,5} + R_{1,2,3,5,6} + R_{1,3,4,5,6} \tag{3.2.3}$$

From (3.1.3) we can see that this is in fact the six-point NMHV tree but let us consider these as three terms in the BCFW expansion of the n-point amplitude. If we make a list of the poles which diverge in two-dimensional kinematics we will now find that there are three terms and each occur in two R-invariants $\langle 3, 4, 5, 1 \rangle$, $\langle 5, 1, 2, 3 \rangle$ and $\langle 3, 5, 6, 1 \rangle$. Clearly at six-points this must be finite since this is the complete amplitude, however since only the same poles occur (and necessarily cancel) independent of how many external points there are, then this must necessarily be a finite combination for R-invariants in two-dimensional kinematics.

Let us take this example and try to reformulate precisely the same argument in a more general way starting with the term $R_{1,i-1,i,j-1,j}$ where we will assume i and j are both even. As such we have two poles which will diverge in the two-dimensional kinematics $\langle 1, i-1, i, j-1 \rangle$ and $\langle j-1, j, 1, i-1 \rangle$. Now just as above there are terms which contain these poles and in turn they also contain an additional but identical pole in each, giving us for example the combination:

$$R_{1,i-1,i,j-1,j} + R_{i-1,j-1,j,1,2} - R_{j-1,1,2,i-1,i} \tag{3.2.4}$$

which should provide us a finite combination and which we have checked numerically. Note that this is not constructed in terms of BCFW-terms with only the form $R_{1,i-1,i,j-1,j}$ but by terms cyclically related to these and given this it is almost unique, we could have started with the same term and completed it as:

$$R_{1,i-1,i,j-1,j} - R_{i-1,j-1,j,n,1} - R_{j-1,n,1,i-1,i} \tag{3.2.5}$$

However these are the only two options if we wish to use BCFW-style terms.

We now wish to show that these are indeed finite combinations independent of the R-invariants in two-dimensional kinematics independently of how many external particles we have. We will then find a nicer way to express these combinations in two-dimensional kinematics which do not have these poles

which diverge term-by-term.

In [8, 92] the authors give an elegant geometric picture for R-invariants as polytopes which has most recently been used and generalise to the Amplituhedron [12, 13]. We will avoid giving all the details of this geometric picture but give the key concepts which will allow us to see that these above combinations do indeed represent finite-combinations of R-invariants in two-dimensional kinematics.

Considering the basic NMHV R-invariants $[a, b, c, d, e]$ and taking an arbitrary deformation $Z_a \to Z_a = zZ_f$ Cauchy's theorem gives the familiar 6-term identity

$$[a,b,c,d,e]+[b,c,d,e,f]+[c,d,e,f,a]+[d,e,f,a,b]+[e,f,a,b,c]+[f,a,b,c,d] = 0$$
$$(3.2.6)$$

This can be seen in the two different representations of the 6-point, tree-level NMHV amplitude (3.1.3). This 6-term identity(3.2.6) is a rearrangement of this. This lack of uniqueness poses a simple question: If we are presented with two different combinations of R-invariants then how do we determine if the two expressions are equivalent. As such we wish to find a method of characterising the equivalence classes of expressions which differ by these identities.

To this end we imagine a "5-simplex" $[abcdef]$ which is completely anti-symmetric in its indices. Then if we consider taking the "boundary" of this simplex using a boundary operator ∂. Each boundary of such a simplex is a "face" of five out of the six arguments with an orientation which meaning that it is completely antisymmetric in its indices. Then our earlier 6-term identities (3.2.6) emerges as

$$\partial\,[abcdef] \quad = [a,b,c,d,e] + [b,c,d,e,f] + [c,d,e,f,a] + [d,e,f,a,b]$$
$$+[e,f,a,b,c] + [f,a,b,c,d] = 0 \quad (3.2.7)$$

What this boundary operator is doing is giving us a democratic list of the faces of this 5-simplex. It is simple to ascertain that $\partial^2 = 0$, which means that R_1 and R_2 are equivalent up to 6-term identities if we can find a simplex σ such that

$$R_1 = R_2 + \partial\sigma \Rightarrow \partial R_1 = \partial R_2 \qquad (3.2.8)$$

As such, whilst the representation of an amplitude in terms of R-invariants is not unique the "boundary" of the amplitude is invariant. This is in fact saying that two sets of R-invariants are equal if the list of their poles is equal,

we obtain these poles in an analogous manner using the same mechanism as above:

$$\partial\,[abcde] = [abcd] + [bcde] + [cdea] + [deab] + [eabc] \qquad (3.2.9)$$

where we note that this produces nothing more than the democratic sum over poles of the R-invariant - this is not merely coincidence and further explanation is given in [8]. To be explicit this result tells us that it is the pole-structure of a set of R-invariants which is the invariant quantity and an amplitude is uniquely determined by its pole structure.

Let us then use this boundary operator (3.2.9) on the combination (3.2.4) to see which poles remain throughout this combination, note that a pole will cancel when two-poles emerge with opposite site e.g. $[a, b, c, d] + [b, c, d, a]$. So if we apply this boundary operator to (3.2.4) and simplify we eventually have the set

$$[i{-}1, i, j{-}1, j] + [i, j{-}1, j, 1] + [j, 1, i{-}1, i] + [j{-}1, j, 1, 2] + [j, 1, 2, i{-}1]$$
$$[2, i{-}1, j{-}1, j] + [1, 2, i{-}1, i] + [2, i{-}1, i, j{-}1] + [i, j{-}1, 1, 2] \qquad (3.2.10)$$

Now we note that since we specified above that i and j were even there are no poles here which vanish in the two-dimensional kinematics. There are however still spurious poles, for example in general the last pole $[i, j{-}1, 1, 2]$ will not correspond to a physical pole of the form $[a{-}1, a, b{-}1, b]$, as such this is NOT a spurious-free combination, it is however a finite combination in generic two-dimensional kinematics.

We are now in a position where we can write down a basis of finite combinations of R-invariants in two-dimensional kinematics. To do this we will work only with R-invariants of the form $R_{r;s,t} = R_{r,s-1,s,t-1,t}$, that is we shall only use R-invariants which are cyclically related to the BCFW-style R-invariants (3.2.1). Given this we find the following cases:

$$R_{r;s,t} + R_{s;t,r} + R_{t;r,s} \qquad\qquad r, s, t \text{ all even/odd} \qquad (3.2.11)$$
$$R_{r;s,t} + R_{s;t,r} - R_{t-1;r,s} \qquad\qquad r, s \text{ even/odd ; } t \text{ odd/even} \qquad (3.2.12)$$
$$R_{r;s,t} - R_{s-1;t,r} - R_{t-1;r,s} \qquad\qquad r \text{ even/odd ; } s, t \text{ odd/even} \qquad (3.2.13)$$

Note that in deriving these combinations the only restrictions we have made are in using BCFW-like terms $R_{r;s,t}$ in order to prevent terms with no physical poles such as $R_{1,4,7,8,9}$ occurring. In particular we have made no restrictions on

the fermionic data (η's or equivalently χ's). We reiterate that these combinations are not spurious-free and the boundary-operator argument (3.2.8) demonstrates that only the full tree-level amplitude can be completely spurious-pole free at any number of external points. However these combinations do form a finite basis in 2d kinematics, we now will turn to taking these combinations and expressing them in a purely two-dimensional fashion which we can use thereafter and which has no disappearing four-brackets to contend with.

3.3 Two-Dimensional R-invariants

We first considered the 6-point, NMHV, tree-amplitude and by hand put in particular χ-components in an effort try to establish a way to write the combination $R_{1,2,3,4,5} + R_{3,4,5,6,1} + R_{5,6,1,2,3}$, that is we simply calculated on a computer the result for particular cases and from this attempted to guess the full answer. In turn we used this to create expressions for the other finite-combinations (3.2.11)-(3.2.13). The first result we found was that there are no non-zero terms which do not have two χ_{even}'s and two χ_{odd}'s, so for example if we look for the χ_1^4-component we must find it is zero. This immediately suggests a factorisation of sorts into the odd-sector and the even-sector. Also, as can only have been the case, the eventual term must always be able to be written in two-brackets of odd-terms and two-brackets of even-terms and these observations in combination led to the definition of a new R-invariant which is a function of three twistors $\tilde{R}(\mathcal{Z}_{r-1}, \mathcal{Z}_r, \mathcal{Z}_{s-1}, \mathcal{Z}_s, \mathcal{Z}_{t-1}, \mathcal{Z}_t)$

$$\tilde{R}(r,s,t) = \frac{\varepsilon_{ABCD} \left(\chi_r \langle s,t \rangle + \text{cycle} \right)^{AB} \left(\chi_{r-1} \langle s-1, t-1 \rangle + \text{cycle} \right)^{CD}}{\langle rs \rangle \langle st \rangle \langle tr \rangle \langle r-1, s-1 \rangle \langle s-1, t-1 \rangle \langle t-1, r-1 \rangle} \quad (3.3.1)$$

for r, s, t all even or odd. Here the indices A, B, C, D in the numerator relate to the fermionic variables so we really mean $\varepsilon_{ABCD} \left(\chi_r^A \langle s,t \rangle + \text{cycle} \right) \left(\chi_r^B \langle s,t \rangle + \text{cycle} \right)$ ... This is a function of six-external particles, three even and three odd and has a natural factorisation into two parts where we may write one part as

$$[a,b,c]^{AB} = \frac{\left(\chi_a \langle bc \rangle + \text{cycle} \right)^{AB}}{\langle ab \rangle \langle bc \rangle \langle ca \rangle} \quad (3.3.2)$$

We briefly remark that in [13] there are dual-triangles which represent amplitudes in a toy-model and these objects (3.3.2) directly correspond to the polytope model for these two-dimensional 'half' R-invariants. We then checked numerically for a variety of different helicity-configurations and for $n \leq 26$ that

the following equations hold:

$$R_{r;st} + R_{s;tr} + R_{t;rs} = -\tilde{R}(r,s,t) \qquad \text{r,s,t all even}$$
$$R_{r;st} + R_{s;tr} - R_{t-1;rs} = \tilde{R}(r,s,t) \qquad \text{r,s even, t odd (or vice versa)}$$
$$R_{r;st} - R_{s-1;tr} - R_{t-1;sr} = -\tilde{R}(r,s,t) \qquad \text{r even, s,t odd (or vice versa)}$$

$$(3.3.3)$$

We have no conclusive justification for why, with the full SU(4)-symmetry intact and as such no restrictions on the χ-variables, we should find that all NMHV remainder functions which are not of the form $\chi_{\text{even}}\chi_{\text{even}}\chi_{\text{odd}}\chi_{\text{odd}}$ vanish. This however is obvious from the right-hand side of (3.3.3) since there we must take two even-χ's and two odd-χ's to achieve an SU(4)-invariant.

We now know that since any collection of R-invariants, at any loop order, must be expressible as a linear combination of our base cases (3.3.3), if they really are to be a basis of finite-terms in 2d kinematics. As such we should be able to express all amplitudes as finite-valued functions multiplied by $\tilde{R}(r,s,t)$ functions or alternatively $[a,b,c]^{AB}$-functions. Note that the first case is more powerful as it tells us that particles appear in pairs $r-1, r$ etc. So we will have terms of the form $[a,b,c]\,[a-1,b+1,c-1]$ etc. This is a good restriction as it enforces some natural form of locality e.g. $\frac{1}{\langle ab \rangle \langle a-1,b+1 \rangle} \sim \frac{1}{\langle a-1,a,b,b+1 \rangle}$ which is a physical pole. However it may become easier subsequently to use the $[a,b,c]^{AB}$ notation.

We find from this the simplest way to express our amplitude (only valid for the restricted $(1+1)$-dimensional case) is:

$$M_{n,1}^{(0)} = \sum_{1<j \triangleleft k<n} \tilde{R}(1,j,k)(-1)^{1+j+k} \qquad (3.3.4)$$

We note that these results provide an improvement on those derived in [46] where following our work the authors ensured that the R-invariants did not blow-up term-by-term by using a reference-twistor and dividing the SU(4) R-symmetry into SU(2) × SU(2) as follows

$$\chi_{\text{even}} = (*, 0, *, 0) \qquad \chi_{\text{odd}} = (0, *, 0, *) \qquad (3.3.5)$$

Where here the asterisks denote non-zero values. As such this results in turning our 'half', two-dimensional R-invariants (3.3.2) into genuine two-dimensional

delta-functions

$$[a, b, c] = \frac{\delta^{0|2}\left(\chi_a\langle bc\rangle + \text{cycle}\right)}{\langle ab\rangle\langle bc\rangle\langle ca\rangle} \tag{3.3.6}$$

However although this varies from our work in the fermionic region, in the bosonic components and at NMHV this completely agrees with our prescription. Once we consider going to N^{2}MHV, differences emerge between the different prescriptions. Were we to write down

$$R_{6,2}^{(0)} = [1, 3, 5]^{AB}\, [1, 3, 5]^{CD}\, [2, 4, 6]^{EF}\, [2, 4, 6]^{GH}\, \varepsilon_{ABEF}\varepsilon_{CDGH} \tag{3.3.7}$$

such an expression would be non-vanishing, whereas with $\text{SU}(4) \rightarrow \text{SU}(2) \times \text{SU}(2)$ one necessarily obtains a $\delta^{0|2}$-function appearing twice in the numerator and as such the amplitude evaluates to zero. So it is clear that beyond NMHV such a prescription loses additional information than merely the bosonic restriction. Note also that this SU(4)-splitting also causes the MHV-prefactor to vanish

$$\delta^{4|4}\left(\sum_{i=1}^{n}\lambda_i\eta_i\right) = \delta^{2|4}\left(\sum_{\text{i even}}\lambda_i\eta_i\right)\delta^{2|4}\left(\sum_{\text{j odd}}\lambda_j\eta_j\right) = 0 \tag{3.3.8}$$

meaning that under this restriction all amplitudes are zero, although the structure of the remainder functions can still be of interest.

Using these insights, as well as the notation derived so far in this chapter, we next attempt to write down the NMHV, one-loop, 8-point amplitude in reduced 2d-kinematics purely from symmetry considerations. We will then consider uplifting to n-points before we introduce the $\overline{Q}$-equation derived in [46] explaining how it relates our one-loop, NMHV amplitude at $(n+2)$-points to the n-point, MHV, 2-loop amplitude, thus providing a non-trivial consistency check.

3.4 ONE-LOOP NMHV AMPLITUDE

We want to build the NMHV one-loop amplitudes in an analogous way to how we constructed the earlier MHV amplitudes - that is, using only symmetries and collinear limits. From our observed result that all NMHV amplitudes have only non-zero contributions for χ-combinations which have two even and two odd χ's, we can say that the form of the NMHV one-loop amplitude will be a sum over terms with an even R-invariant part, an odd R-invariant part and a

weight-two polylogarithm with cross-ratio arguments. These two R-invariant halves are naturally contracted by the completely antisymmetric tensor ϵ_{ABCD} but we will not write this out explicitly. To use an analogous argument to the MHV case we will first have to develop a notation which allows us to explicitly see which collinear limits are being imposed by the R-invariants.

To this end we return to

$$[a, b, c]^{AB} = \frac{\delta^{AB}(\chi_a\langle bc\rangle + \text{cyclic})^{AB}}{\langle ab\rangle\langle bc\rangle\langle ca\rangle} \tag{3.4.1}$$

and under the limits of colliding any two of these points it is simple to see they vanish e.g.

$$\lim_{a\to b}[a, b, c]^{AB} = \frac{\delta^{AB}(\chi_b\langle bc\rangle + \chi_b\langle cb\rangle)}{\langle bb\rangle\langle bc\rangle\langle cb\rangle} = 0 \tag{3.4.2}$$

As such this term vanishes in the limits $a \leftrightarrow b$, $b \leftrightarrow c$ and $c \leftrightarrow a$. If on the other hand we take a pair

$$[a, b, c]^{AB} - [a, b, d]^{AB} = \frac{\delta^{AB}(\chi_a\langle bc\rangle + \text{cycle})}{\langle ab\rangle\langle bc\rangle\langle ca\rangle} - \frac{\delta^{AB}(\chi_a\langle bd\rangle + \text{cycle})}{\langle ab\rangle\langle bd\rangle\langle da\rangle} \tag{3.4.3}$$

this now vanishes under two collinear limits, both terms disappearing individually under $a \leftrightarrow b$ and the terms cancelling one another under the limit $c \leftrightarrow d$. This combination will be more useful for our purposes.

We use the notation

$$M_{i,j} = [i{-}1, i{+}1, j{-}1] - [i{-}1, i{+}1, j{+}1] \tag{3.4.4}$$

and thus imposes triple collinear limits:

$$\lim_{i-1\leftrightarrow i+1} M_{i,j} = 0 = \lim_{j-1\leftrightarrow j+1} M_{i,j} \tag{3.4.5}$$

note that 'i' and 'j' must be both even or both odd. There is a special case in this definition where $j = i \pm 2$, for example if $j = i + 2$ then we have only one term in our $M_{i,i+2} = -[i{-}1, i{+}1, i{+}3]$, however this still imposes the triple collinear limits across i and $i{+}2$. We might legitimately ask if there is not some other linear combination of R-invariants which imposes 3 collinear limits? Of course a simple 3-bracket does do this but they will not all be triple collinear limits with the maximum being two as in $M_{i,i+2}$. If we attempt to add more terms to impose more triple-collinear limits we find that we cannot and our

equations always reduce to a simple $M_{i,j}$ form due to the linear restriction:

$$[a, b, c] - [b, c, d] + [c, d, a] - [d, a, b] = 0 \qquad (3.4.6)$$

this equation also tells us $M_{i,j} = -M_{j,i}$.

In analogy to these new R-invariant combinations we will also introduce a compact form for writing our the logarithmic functions of cross-ratios:

$$L_{i,j} = \log(u_{i,j}) \qquad (3.4.7)$$

We now see that both $L_{i,j}$ and $M_{i,j}$ disappear in the triple-collinear limits across i and j (2.9.6). As such we now find that M's and L's play very similar roles in the construction of our amplitudes and this similarity is not accidental as we will show at the end of this chapter when we relate the one-loop NMHV amplitude to the two-loop MHV amplitude, using the $\bar{Q}$-equation [46].

Before this though let us build the NMHV one-loop amplitude which must be of the form

$$\sum M_{\text{odd}} M_{\text{even}} (L \otimes L) \qquad (3.4.8)$$

Where this tensor product is that of the symbol defined in the previous chapter. Since each M can impose two triple-collinear limits, then the tensor product must implement two even and two-odd limits, since we want to implement at least 8 different collinear limits meaning that we cannot obtain a non-trivial 6-point contribution. Note that $u_{i,i+2} = 0$ (2.7.12) and this means that we are unable use $M_{i,i+2}$ in the construction of a term which vanishes under all triple-collinear limits since we would have be unable to put a term in the symbol which imposes the other two collinear limits across $i + 4$ and $i + 6$. As such at 8-points, largely in analogy with the MHV derivation from Sect 2.9, we find we have only one term which satisfies the restrictions of collinear limits ((2.9.6) and (3.4.5)), cyclicity and parity:

$$\alpha M_{1,5} M_{2,6} \left(L_{3,7} \otimes L_{4,8} + L_{4,8} \otimes L_{3,7} \right) + \text{cyclic} = M_{1,5} M_{2,6} \log(u_{3,7}) \log(u_{4,8}) + \text{cyclic} \qquad (3.4.9)$$

Once again we are impressed with the simplicity of this equation and this alone helps to motivate further exploration into reduced kinematic amplitudes to higher k, l and n.

If we look to the 10-point case then in much the same vein as the two-loop MHV case, we can see that we cannot construct any term which vanishes in all 10 triple collinear limits. As such we again need only consider how to uplift

the 8-point remainder function to 10-points so that it reduces to the correct 8-point function in all triple-collinear limits. The uplifting takes precisely the same form as the two-loop MHV case (2.9.20)-(2.9.21):

$$S_{8,1}^{(1)}(i,j,k,l) = R_{8,1}^{(1)}(i{-}1, i{+}1, j{-}1, j{+}1, k{-}1, k{+}1, l{-}1, l{+}1)$$
$$\mathcal{R}_{n,1}^{(1)} = \sum_{i\lhd j\lhd k\lhd l}(-1)^{i+j+k+l}S_{8,1}^{(1)}(i,j,k,l) + \text{constant} \qquad (3.4.10)$$

This identical uplift, along with the similarity between the forms of the 8-point MHV 2-loop remainder function (2.9.14) and the 8-point NMHV one-loop function (3.4.9), where we will repeat the 2-loop MHV uplifting formula here to allow easier comparison between the two forms.

$$S_{8,0}^{(2)}(x_i, x_j, x_k, x_l) = \mathcal{R}_{8,0}^{(2)}(i{-}1, i, j{-}1, j, k{-}1, k, l{-}1, l)$$
$$\mathcal{R}_{n,0}^{(2)} = \sum_{i\lhd j\lhd k\lhd l\lhd i} S_{8,0}^{(2)}(x_i, x_j, x_k, x_l) + \text{constant} \qquad (3.4.11)$$

This led us to attempt to generalise the process of building the functions stratified by how many triple-collinear limits they disappear under, as well as an uplifting process for all n-point, N^{k}MHV, ℓ-loop amplitudes and this is the topic of the next chapter. First however, we will introduce one last tool developed in [46] which directly links the NMHV one-loop amplitude to the two-loop MHV amplitude and further justifies the search for our universal uplift.

3.5 $\bar{Q}$-EQUATION

The BDS subtracted amplitude must be invariant under a chiral half of the dual superconformal symmetry [33, 45, 100] as well as the R-symmetry. Indeed it is believed to be invariant under dual-conformal operators as motivated by the dualities with Wilson Loops [34, 44, 61, 99] and correlation functions [67, 68, 73]. So the remainder function is left unchanged under action by the following operators:

$$Q_A^a = (Q_A^\alpha, \bar{S}_A^{\dot\alpha}) := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial \chi_i^A} \qquad R_B^A := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial \chi_i^B} \qquad K_b^a := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial Z_i^b}$$
$$(3.5.1)$$

However the BDS-subtracted amplitude does not a priori behave equivalently under generators of the other chiral half,

$$\bar{Q}_a^A = (S_a^A, \bar{Q}_{\dot{\alpha}}^A) := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a} \qquad (3.5.2)$$

In [42, 46] it is conjectured that this symmetry can however be restored through a quantum corrected $\bar{Q}$-operator, defined by the following equation [46] in the full four-dimensional theory

$$\bar{Q}_a^A R_{n,k} = a \operatorname{Res}_{\varepsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3} \mathcal{Z}_{n+1})_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\mathrm{tree}} \right] + \text{cyclic} \quad (3.5.3)$$

Where we take the residue at $\varepsilon = 0$ of this integral. Here $a = 1, 2, 3, 4$ is a momentum twistor index, A is the SU(4)-index and ε, τ parametrise $\mathcal{Z}_{n+1}$ in the collinear limit (τ being related to the longitudinal momentum fraction)

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \varepsilon \mathcal{Z}_{n-1} + C \varepsilon \tau \mathcal{Z}_1 + C' \varepsilon^2 \mathcal{Z}_2 \qquad (3.5.4)$$

and a as an overall factor being $a = a(g^2)$, one quarter of the anomalous cusp dimension

$$a := \frac{1}{4} \Gamma_{\mathrm{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \cdots \qquad (3.5.5)$$

which is known exactly for all values of the coupling.

This quantum corrected operator is well worth investigation purely on its own merits, however here we will be shortly giving a much simpler two-dimensional version for use on our equations and as such we will try to keep details of the full version limited. The equation is anticipated to be exact at all orders of the coupling, though in [46] it was only approached perturbatively in a. In this paper (3.5.3) was proposed to be an all-loop expression for the action of the dual-superconformal generators $\bar{Q}$, in terms of a one-dimensional integral over the collinear limit of a higher-point amplitude. We will write the simpler two-dimensional version of the $\bar{Q}$-equation (3.5.3):

$$
\begin{aligned}
\bar{Q}_a^A R_{2n,k} \;&=\; a \int d^{1|2} \lambda_{n+1}^+ \int d^{0|1} \lambda_{n+1}^- \left(R_{2n+2,k+1} - R_{2n+2,1}^{\mathrm{tree}} R_{2n,k} \right) + \text{cyclic} \quad (3.5.6) \\
&:= a \lambda_{n,a}^- \lim_{\lambda_{n+1}^- \to \lambda_n^-} \int_{\lambda_n^+}^{\lambda_1^+} \langle \lambda_{n+1}^+ d\lambda_{n+1}^+ \rangle \int d^2 \chi_{n+1}^+ (d\chi_{n+1}^-)^A (\text{parenthesis}) \\
&\quad + \text{cyclic}
\end{aligned}
$$

However we will rewrite this later in a more user-friendly format.

The simplified setting of two-dimensional kinematics was a problem approached in [46] which used our idea of splitting the SU(4) symmetry into SU(2) × SU(2), and showed how we realte tree-level N^{2}MHV → one-loop NMHV → two-loop MHV in these kinematics through (3.5.6). We have used these ideas often in our exploration of amplitudes in two-dimensional kinematics and have developed a more compact way of enacting the process by which the right-hand side of (3.5.6) can be simplified. As such in the remainder of this chapter we will outline our notation for this equation and use it to show how to relate the 1-loop NMHV to the 2-loop MHV amplitude.

We express the $\bar{Q}$-equation in the following manner:

$$\bar{Q}_n R_{n,k}^\ell = \lim_{n+2\to n} Q_{n+2} \left(\lim_{n+1\to n-1} - \lim_{n+1\to 1} \right) \mathcal{O}_{n+1} R_{n+2,k+1}^{\ell-1} + \text{cyclic} \qquad (3.5.7)$$

We define the operations Q, $\bar{Q}$ and $\mathcal{O}$ as:

$$\mathcal{O}_{i,AB} = \frac{\partial^2}{\partial \chi_i^{AB} \partial \chi_i^B \int \langle Z_i dZ_i \rangle}$$

$$Q_A = \sum_{i=1}^n Q_{i,A} \qquad Q_{i,A} = Z_i \frac{\partial}{\partial \chi_i^A}$$

$$\bar{Q}^A = \sum_{i=1}^n \bar{Q}_i^A \qquad \bar{Q}_i^A = \chi_i^A \frac{\partial}{\partial Z_i} \qquad (3.5.8)$$

the operations Q and $\mathcal{O}$ have the following effects (where $M_{i,j;k,l} = [i,j,k] - [i,j,l]$):

$$\mathcal{O}_{n+1} M_{i,j;k,n+1} \to L_{i,j;k,n+1}$$

$$Q_n [i,j,n+2] = \bar{Q}_n \log \frac{[n+2,j]}{[n+2,i]} \qquad (3.5.9)$$

As such, for our case of interest we are to consider the following equation where we shall input the right-hand side to obtain known expressions for the left-hand side.

$$\bar{Q}_8 R_{8,0}^{(2)}(1,2,\ldots,8) = (\lim_{9\to7} - \lim_{9\to1})\mathcal{O}_9 \lim_{10\to8} Q_{10} R_{10,1}^{(1)}(1,2,\ldots,10) \qquad (3.5.10)$$

Knowing the form of the uplift in terms of S's given in (3.4.10) we can immediately evaluate this equation. However rather than consider all possible S's it is simple to see we need only those which contain both 10 and 9 as well as not disappearing in the collinear limit $\lim_{10\to8}$ alongside one of $\lim_{9\to7}$ or $\lim_{9\to1}$. As such, it is sufficient to consider only the pair $S_8(x_2, x_4, x_6, x_{10}) -$

$S_8(x_3, x_5, x_7, x_{10})$. Now let us take just the first term here (2.9.20), defined as

$$
\begin{aligned}
S_8(x_2, x_4, x_6, x_{10}) \;\; &= (M_{1,3;5,9} L_{3,5;9,1} + M_{3,5;9,1} L_{1,3;5,9}) \\
&\times (M_{2,4;6,10} L_{4,6;10,2} + M_{4,6;10,2} L_{2,4;6,10}) \quad (3.5.11)
\end{aligned}
$$

which can be seen to transform under the right-hand side of (3.5.7) to:

$$
2 L_{1,3;5,7} L_{3,5;7,1} \left(L_{4,6;8,2} \bar{Q}_8 \log \frac{[8,4]}{[8,2]} + L_{2,4;6,8} \bar{Q}_8 \log \frac{[8,4]}{[8,6]} \right) \quad (3.5.12)
$$

equally the other term $S_8(x_3, x_5, x_7, x_{10})$ evaluates to precisely the same thing and as such we simply need to add cyclic terms as in (3.5.7). Note at this stage however the dual-conformal invariance is not manifest in our writing although we know it must be present as the left-hand side of (3.5.7) is manifestly dual-superconformally invariant.

The fact that this intermediate stage 'loses' a symmetry, which is necessarily present at both the start and end of the calculation, is a strong clue that there must be a better way to perform this calculation. Such a rewriting would not require the symmetry to be magically reassembled during the last stage of cycling the terms. However despite much effort we have failed to find such a way to enforce this despite the very obvious similarities between the starting point and the final equation for the simplest examples such as this. If we finish off now by cycling our answer, we find that up to an overall integer factor:

$$
\bar{Q}_8 R_{8,0}^{(2)} = \bar{Q}_8 \left(L_{1,3;5,7} L_{3,5;7,1} L_{2,4;6,8} L_{4,6;8,2} \right) \quad (3.5.13)
$$

as we would of course expect. The constant of proportionality relates the unfixed constant (from our method) for the 1-loop NMHV amplitude to that of the 2-loop MHV amplitude. Note also the striking and suggestive result that to relate the 1-loop NMHV result to the 2-loop MHV result is as simple as setting $M \to L$ throughout the NMHV amplitude and multiplying by the correct constant factor. Despite this very suggestive approach we have found that this naive approach does not extend in full generality beyond these early examples.

The $\bar{Q}$-equation does provide restrictions distinct from those mentioned earlier at sufficiently high complexity, such as on the NMHV, 2-loop amplitude by relating it to the 3-loop MHV amplitude. However, we have not found a concise way to see these restrictions beyond simply following through the entire calculation with unfixed coefficients and seeing what restrictions must take

place to allow for the dual-conformal symmetry to be reunited. In conclusion it would be very desirable to find a re-casting of this expression which maintains dual-conformal invariance at all stages however for the moment such a re-casting is not available. We will refer to this equation (3.5.7) in the next chapter which is devoted to generalising the S-equation uplifting mechanism to general amplitudes in 2-dimensional kinematics.

4

UPLIFTING AMPLITUDES IN TWO - DIMENSIONAL KINEMATICS

In this chapter we will first demonstrate the manner in which we build the general form of the 10-point ℓ-loop MHV amplitude written in terms of the 8-point amplitude and we provide explicit expressions at 2- and 3-loops. We subsequently use this to motivate our main result of the chapter: firstly a general n-point, ℓ-loop collinear uplift formula for MHV amplitudes followed by a generalisation of this to N^kMHV amplitudes. We outline the work which has been done since the publication of these equations in [80] and discuss what specialist redundancy is needed so that tree-level NMHV amplitudes are not included in this uplifting process. This chapter is based primarily on work in [80] with most subsections mapping directly across, there are however updates to maintain consistency with more recent work.

4.1   8- AND 10-POINT MHV AMPLITUDES AT 3 LOOPS

The 8-point MHV amplitude at 3 loops $\mathcal{R}_8^{(3)}$ was first determined in [91], where this derivation was based on the assumption that the amplitude has a symbol whose entries are cross-ratios[2]. This assumption has recently been revised so that at 3-loops we now have additional symbol entries such as u_1-u_2 and $1-u_1-u_2$ (2.7.15), however such alterations do not effect the uplifting construction but merely the form of the components. We review the construction of the

---

[2]   At 1 and 2-loops this amounts to logarithms only in the amplitude [91] and starting from 3-loops the reconstructed functions involve also polylogarithms $\mathrm{Li}_n(\cdot)$.

uplifting formula as presented in [80] and the reader is referred to Appendix.A for additional details on the construction of the component functions.

Insisting that the 8-point function be cyclically (and parity) symmetric, and that it vanishes in the collinear limit (2.9.6) $z_8 \to z_6$ i.e. $u_{1,5} \to 0$, $u_{3,7} \to 1$ with $u_{2,6}, u_{4,8}$ unconstrained [3] leads to a 3-loop amplitude of the form:

$$\mathcal{R}_8^{(3)} = \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma(u_1) f_\tau(u_2) \tag{4.1.1}$$

Here $a_{\sigma\tau} = a_{\tau\sigma}$ are some rational coefficients, and the sum is over the set of functions f_σ with the following properties:

$$f_\sigma(u) = f_\sigma(1-u) \tag{4.1.2}$$

$$f_\sigma(0) = 0 \tag{4.1.3}$$

$$f_\sigma(u) \text{ is a (generalised) polylogarithm.} \tag{4.1.4}$$

(4.1.2) is required for reasons of cyclic symmetry, since under a cyclic shift by two: $u_{1,5} \to u_{3,7} = 1 - u_{1,5}$. Whereas (4.1.3) is required, since alongside (4.1.2) it imposes the triple-collinear limit restrictions (2.9.6). Furthermore the total polylog weight (or "degree of transcendentality") of $\mathcal{R}_8^{(3)}$ must be six, due to the uniform transcendentality property of perturbative amplitudes in $\mathcal{N}{=}4$ SYM. Indeed in general this means that at ℓ-loop order the polylog weight is 2ℓ for all functions.

In [91] all possible functions f_σ with only simple cross-ratios in the symbol were listed (see also Appendix A where these functions are called f_σ^+), and there we also take the additional terms with symbol entries of the form $u_1 - u_2$ etc. from [47]. In [91] there is a unique weight-two function $f(u) = \log(u)\log(1-u)$, 3 weight-three functions, and 7 weight-four functions, leading to a total of 13 a priori unfixed coefficients $a_{\sigma\tau}$. Further constraints arise from the OPE analysis of [77] which fix 6 of these, leaving 7 unfixed coefficients[4].

The form of the 8-point amplitude (4.1.1) generalises straightforwardly beyond three-loops by simply allowing the functions f_σ to have more general

---

[3] Note that we need not consider cyclically equivalent collinear limits $z_i \to z_{i-2}$, since they will follow automatically from cyclic symmetry.

[4] It is tempting to assume a further simplification of the structure, namely that the f_σ are of weight 3 only. This would be consistent with all currently known facts and would leave just 3 unfixed coefficients, however we will not be making this assumption in this work.

weight, such that the total weight is 2ℓ.

$$R_8^{(\ell)} = \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_2) \qquad (4.1.5)$$

It is also valid at two-loops where there is only one allowed function (up to a multiplicative constant), $f^{(2)}(u) = \log(u)\log(1-u)$, and we reproduced the original two-loop result at 8-points found in [62] and reviewed in Chapter 2

$$R_8^{(2)} = -\frac{1}{2}\log(u_1)\log(u_2)\log(u_3)\log(u_4) + \text{constant} . \qquad (4.1.6)$$

We recast the uplift which we already have, of the 8-point amplitude to 10-points which we presented in Chapter 2 for the 2-loop, MHV case (2.9.21) and in Chapter 3 for the 1-loop NMHV amplitude (3.4.10), and following [91] write it in a more general fashion which permits a 10-point vanishing contribution. The idea being to write down all 10-point functions which reduce to the 8-point amplitude in the triple collinear limit, plus an additional contribution which is required to vanish in all such limits. This lead to [5]

$$R_{10}^{(\ell)} = \ \frac{1}{2}\sum_{\sigma,\tau} a_{\sigma\tau} \left(f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_2) - f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_4) + \frac{1}{2} f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_6) \right)$$
$$+\text{cyclic}+V_{10}^{(\ell)}$$

$$(4.1.7)$$

The last term $\left(V_{10}^{(\ell)} \right)$ denotes a generic 10-point function which vanishes in all triple collinear limits. It is reproduced in Appendix A at 3-loop level from Ref. [47, 91]

The construction of the non-vanishing contribution under triple collinear limits (everything apart from V_{10}) was specific to the case at hand where the 10-point amplitude reduces to the 8-point amplitude. If we want to uplift (4.1.7) to 12 and higher points, we need to come up with an alternative, generalised and potentially more geometric procedure.

Note that the general 10-point expression has a more complicated structure than the result at two loops:

$$\tilde{R}_{10}^{(2)} = -\frac{1}{2}\left(\log(u_1)\log(u_2)\log(u_3)\log(u_4)\right) + \text{cyclic} . \qquad (4.1.8)$$

---

[5] The original derivation in [91] was performed at 3-loops, but the resulting expression (in terms of the functions $f^{(\ell)}$) remains valid at ℓ-loops.

The reason for this simplification is the corresponding simple form of the 2-loop function $f^{(2)}(u) = \log(u)\log(1-u)$. Using this, together with the fact that at 10 points $1-u_1 = u_9 u_3$ (2.7.15) along with cyclic symmetry, one can check that all the minus signs in (4.1.7) disappear (for $\ell = 2$) and the result reduces to (4.1.8).

It is becoming clear that at the level of the symbol it is extremely useful to cast things in terms of the fundamental cross-ratios u_{ij} since they (or more precisely their logarithms) form a basis for the vector space upon which the tensor-symbol is constructed. However as far as writing functions down at higher points, these are not the natural objects to use. For example we can see that the function $f(u_1) = f(1 - u_1) = f(u_9 u_3)$ can be thought of as a function of the product

$$u_9 u_3 = u_{10\,6;4\,2} \tag{4.1.9}$$

which is not a simple cross-ratio.

At a higher number of points there is no reason to expect only functions of the simple cross-ratios to appear. There is of course no contradiction here as even though the function in our example has an argument which is a product of simple cross-ratios, the symbol of this function will be expressed in terms of the simple cross-ratios themselves due to the product property of the symbol (2.7.16). Thus the symbolic construction of amplitudes in [91], based on the identification of the basis of the amplitude symbol in terms of simple u_{ij}'s continues to hold in this way, but the functions corresponding to these symbols at higher n are not best represented in terms of *simple* cross-ratios. For our example (4.1.9) it is clear that the natural argument would be the cross-ratio $u_{10,6;4,2}$. Note that we still have to relax this condition as it remains impossible to express $u_{1,5} - u_{2,6}$ as a product of simple cross-ratios. However, the form of these extra contributions is heavily limited, as briefly addressed in Sect 2.9 in our derivation of the 8-point, 2-loop MHV amplitude, and these additional terms do not influence the derivation of the uplifting formula since they do not effect the number of collinear limits a term may disappear under.

Let us consider recasting $\tilde{R}_{10}^{(\ell)}$ in terms of the *complete* 8-point amplitudes $\tilde{R}_8^{(\ell)}$ as we have already seen in earlier chapters, rather than manipulating its building blocks $f_{\sigma/\tau}(u)$ as was done in (4.1.7). We are in fact able to completely solve the constraints from collinearity using this process.

4.2   8− AND 10−POINTS RECAST

We can completely and explicitly solve the constraints coming from collinear limits at 3-loops in terms of three structures, related to the 8-,10- and 12-point amplitudes and more generally at ℓ-loops in terms of the m-point functions S_m with $m \leq 4\ell$. But first, to motivate the general formula, we recast the 8- and 10-point amplitudes in a form more suitable for generalisation, and in the process introduce the new concepts we will need.

In the two examples we have given thus far (2.9.21) and (3.4.10) we defined an intermediary function S_8 (2.9.20). This was a function of x_i's related to the 8-point amplitude, and we wrote the uplift in terms of this object. We will write all our uplift formulae in terms of S-functions and for the moment we will keep relating the 8-point functions S_8 directly with the amplitudes $\tilde{R}_8$ as indeed in earlier chapters. In the following subsection we will argue that this is a particularly simple example of a more general procedure and we will rewrite the S-functions accordingly.

Our first step is to return the problem back to a function of z's, that is,

$$R_{8,1}^{(2)}(u_1, u_2) = R_{8,1}^{(2)}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \ . \tag{4.2.1}$$

Now, in making attempts to lift this to higher points, we notice that in the higher point amplitudes which are combinations of 8-point functions as in (2.9.21), the z's always appear in consecutive pairs, but with the odd element of the pair always appearing before the even element. This is exactly what happens in the definition of x_i in terms of z_i:

$$p_i^{\alpha\dot\alpha} \equiv \lambda_i^\alpha \lambda_i^{\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} \qquad \alpha, \dot\alpha = 1, 2 \tag{4.2.2}$$

It suggests that we further think of the amplitude as a function of position coordinates x's instead of z's so that:

$$2S_8^{(2)}(x_i, x_j, x_k, x_l) := \tilde{R}_8^{(2)}(x_i^+, x_i^-, x_j^+, x_j^-, x_k^+, x_k^-, x_l^+, x_l^-) \ , \tag{4.2.3}$$

Where we defined $\tilde{R}$ in (2.8.2), but which we repeat here

$$\tilde{R}_n = R_n - \frac{n-4}{2} R_6 \ , \tag{4.2.4}$$

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(4.2.3) implies,

$$2S_8(x_2, x_4, x_6, x_8) = \tilde{R}_8(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \ . \qquad (4.2.5)$$

These equations can be viewed as defining the function S_8.[6] Thus via S_8 we are specifying the Wilson loop zig-zag contour (see Fig.1 in Chapter 2) by specifying every second vertex. To allow us a better insight into the consequences of this decomposition, let us examine the symmetries of the function $S_8(x_2, x_4, x_6, x_8)$. The symmetries of the Wilson loop $\tilde{R}_8(z_1, \dots z_8)$, namely cyclic symmetry C_n under which each $z_i \rightarrow z_{i+1}$, and parity of the 8-point Wilson loop $\tilde{R}_8(z_1, \dots z_8) \rightarrow \tilde{R}_8(z_8, \dots z_1)$ give the following

$$S_8(x_2, x_4, x_6, x_8) = S_8(x_4, x_6, x_8, x_2) = S_8(x_6, x_8, x_2, x_4) = S_8(x_8, x_2, x_4, x_6)$$
$$S_8(x_2, x_4, x_6, x_8) = S_8(x_1^f, x_3^f, x_5^f, x_7^f)$$
$$S_8(x_2, x_4, x_6, x_8) = S_8(x_8^f, x_6^f, x_4^f, x_2^f) \ . \qquad (4.2.6)$$

The first equation follows from cyclicity in z applied twice, i.e. $z_i \rightarrow z_{i+2}$, and the second equation is a consequence of $z_i \rightarrow z_{i+1}$. In the last two equations we have defined the flipped x position

$$x = (x^+, x^-) \quad \Rightarrow \quad x^f = (x^-, x^+) \ . \qquad (4.2.7)$$

This is necessary in order to properly define the cyclic symmetry in terms of the x-variables.

Interestingly, for the 8-point amplitude in the form (4.1.5) there exists this additional discrete symmetry – the flip symmetry – where each x-argument of S_8 becomes flipped,

$$S_8^{(\ell)}(x_i, x_j, x_k, x_l) = S_8^{(\ell)}(x_i^f, x_j^f, x_k^f, x_l^f) \qquad (4.2.8)$$

despite the fact that this is not an expected symmetry of the Wilson loop contour. We identify this symmetry by considering S_8 of even arguments,

$$2S_8(x_2, x_4, x_6, x_8) = \tilde{R}_8^{(\ell)}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = \sum_{\sigma,\tau} a_{\sigma\tau}^{(\ell)} f_\sigma^{(\ell)}(u_1) f_\tau^{(\ell)}(u_2)$$

$$(4.2.9)$$

---

[6] In the following section we will in fact generalise this definition by including an additional contribution to S_8 which is distinct from the 8-point amplitude $\tilde{R}_8$ but will vanish in the 8-point combination and only emerge at higher-loops.

and compare this with S_8 written in terms of the same variables having been flipped,

$$2S_8(x_2^f, x_4^f, x_6^f, x_8^f) \; = \; \tilde{R}_8(z_2, z_1, z_4, z_3, z_6, z_5, z_8, z_7) \; = \; \sum_{\sigma, \tau} a_{\sigma \tau} f_\sigma(u_2) f_\tau(u_1)$$

$$(4.2.10)$$

To understand the right-hand side, note that cross-ratios $u_1 = u_{15}$ and $u_2 = u_{26}$ by definition depend only on even or on odd z-variables respectively,

$$u_1 \; = \; \frac{\langle 86 \rangle \langle 24 \rangle}{\langle 84 \rangle \langle 26 \rangle} = \frac{z_{86} z_{24}}{z_{84} z_{26}}, \qquad u_2 \; = \; \frac{\langle 17 \rangle \langle 35 \rangle}{\langle 15 \rangle \langle 37 \rangle} = \frac{z_{17} z_{35}}{z_{15} z_{37}}, \qquad (4.2.11)$$

hence the distribution of z_i's inside $\tilde{R}_8$ in (4.2.9,4.2.10) implies that these two equations are related by $u_1 \leftrightarrow u_2$. The symmetry $a_{\sigma \tau} = a_{\tau \sigma}$ alongside the summation over all functions f_σ and f_τ implies that the resulting expressions are symmetric under the interchange $u_1 \leftrightarrow u_2$ and equation (4.2.8) follows.

Note also that S_8 satisfies the following properties under the collinear limit $z_8 \to z_6$ (i.e. $x_8 \to x_7$ as can be seen immediately from Fig.1 in Chapter 2):

$$\lim_{x_8 \to x_7} S_8(x_2, x_4, x_6, x_8) = 0 \;, \qquad (4.2.12)$$

or more generally/geometrically

$$S_8(x_i, x_j, x_k, x_l) = 0 \qquad \text{if} \qquad x_k \text{ and } x_l \text{ are light-like separated .} \quad (4.2.13)$$

Having defined the object S_8 we now re-examine the 10-point, ℓ-loop amplitude (4.1.7):

$$\tilde{R}_{10}^{(\ell)} = \; \tfrac{1}{4} \sum_{\sigma, \tau} a_{\sigma \tau} \, f_\sigma^{(\ell)}(u_1) \left(f_\tau^{(\ell)}(u_2) - f_\tau^{(\ell)}(u_4) + f_\tau^{(\ell)}(u_6) - f_\tau^{(\ell)}(u_8) + f_\tau^{(\ell)}(u_{10}) \right)$$

$$+ \text{cyclic} + V_{10}^{(\ell)} \qquad (4.2.14)$$

This can be rewritten in terms of S_8 in a suggestive way which will allow generalisation to n-points as

$$\tilde{R}_{10}^{(\ell)}(z_1, z_2, \ldots, z_8) = \sum_{1 \leq i_1 \lhd i_2 \lhd i_3 \lhd i_4 \leq 10} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1 + \cdots + i_4} \; + V_{10} \;.$$

$$(4.2.15)$$

where, as before, V_{10} is an additional collinear vanishing contribution. The summation convention in this formula has been used a few times, we recall to the reader that it simply states that each $i_k > i_{k-1} + 1$.

The alternating sign in the sum in this formula combined with the property (4.2.13) of S_8 are enough to show that this has the right behaviour under collinear limits and we will see this explicitly below. However more interestingly these observations lead to immediate generalisation to higher points and arbitrary loop order. Thus far we have done no more than generalise to ℓ-loops the uplifting procedure from $8 \to 10$ points which we used in earlier chapters, motivating this uplifting procedure from our perspective of collinear limits more explicitly. However next we produce our first key result of the chapter: the full MHV uplift formula.

4.3 THE GENERAL FORMULA FOR THE n-POINT COLLINEAR UPLIFT

We claim that the n-point MHV amplitude for $\ell \geq 1$, at any loop order is given by

$$
\begin{aligned}
\tilde{R}_n^{(\ell)}(z_1, z_2, \ldots, z_n) = & \sum_{1 \leq i_1 \lhd i_2 \lhd i_3 \lhd i_4 \leq n} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1 + \cdots + i_4} + \\
& + \sum_{1 \leq i_1 \lhd \cdots \lhd i_5 \leq n} S_{10}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1 + \cdots + i_5} + \\
& + \sum_{1 \leq i_1 \lhd \cdots \lhd i6 \leq n} S_{12}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1 + \cdots + i_6} + \\
& + \cdots + \\
& + \sum_{1 \leq i_1 \lhd \cdots \lhd i_{2\ell} \leq n} S_{4\ell}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_{2\ell}})(-1)^{i_1 + \cdots + i_{2\ell}} \, . \quad (4.3.1)
\end{aligned}
$$

Here in order to simplify the notation we have defined the symbol $\lhd$ as follows $i \lhd j \Leftrightarrow i < j - 1$. This operation removes terms in the sum with consecutive x's e.g. $S_m(\ldots, x_i, x_{i+1}, \ldots)$.

This is a deceptively simple formula. The full n-point amplitude for arbitrary n, and arbitrary loop order is given explicitly in terms of just $(2\ell - 3)$ m-point functions, S_m with $m = 8, 10, 12, \ldots 4\ell$. Let us start by returning to the minimal case of $n = 8$ external particles. Equation (4.3.1) then implies,

$$
\tilde{R}_8(z_1, z_2, \ldots, z_8) = S_8(x_2, x_4, x_6, x_8) + S_8(x_1, x_3, x_5, x_7) \, . \quad (4.3.2)
$$

A simple possibility is that the two terms are in fact the same, $S_8(x_2, x_4, x_6, x_8) = S_8(x_1, x_3, x_5, x_7) = \frac{1}{2}\tilde{R}_8$ which is the method we previously employed in chapter two when we found the MHV 2-loop amplitude and the NMHV 1-loop amplitude.

There is however, a more general solution to this equation, necessary beyond 2-loops, where $S_8(x_2, x_4, x_6, x_8) \neq S_8(x_1, x_3, x_5, x_7)$. To examine it, we rewrite (4.3.2) in terms of z-variables,

$$\tilde{R}_8(z_1, z_2, \dots, z_7, z_8) \;=\; S_8(z_1, z_2, \dots, z_7, z_8) + S_8(z_1, z_8, \dots, z_7, z_6)\,. \quad (4.3.3)$$

The left-hand side must by cyclically symmetric in $z_i \to z_{i-1}$. To guarantee this we must impose the flip symmetry (4.2.8) on S_8. When applied to the second term on the right-hand side of (4.3.3) we find,

$$\tilde{R}_8(z_1, z_2, \dots, z_7, z_8) \;=\; S_8(z_1, z_2, \dots, z_7, z_8) + S_8(z_8, z_1, \dots, z_6, z_7)\,, \quad (4.3.4)$$

which automatically gives a cyclically symmetric combination, even though the S_8 individually are not required to have it. We can now divide S_8 into two parts, so that,

$$S_8(x_2, x_4, x_6, x_8) \;=\; \frac{1}{2}\mathcal{R}_8(z_1, z_2, \dots, z_8) + T_8(x_2, x_4, x_6, x_8)\,, \quad (4.3.5)$$

$$S_8(x_1, x_3, x_5, x_7) \;=\; \frac{1}{2}\mathcal{R}_8(z_1, z_2, \dots, z_8) + T_8(x_1, x_3, x_5, x_7)\,. \quad (4.3.6)$$

T_8 is thus an additional contribution to S_8, which is not determined by the amplitude $\mathcal{R}_8$. To ensure that T_8's indeed do not appear in (4.3.2) we require that

$$T_8(x_2, x_4, x_6, x_8) + T_8(x_1, x_3, x_5, x_7) \;=\; 0\,, \quad (4.3.7)$$

which is guaranteed by the flip symmetry of T_8 together with the *anti*-symmetry under $z_i \to z_{i+1}$,

$$T_8(x_1, x_3, x_5, x_7) \to T_8(x_1^f, x_3^f, x_5^f, x_7^f) = -T_8(x_2, x_4, x_6, x_8)\,. \quad (4.3.8)$$

The entire S_8 can be constructed using the method of [91] as we explain in Appendix A. In other words, the contributions to the amplitude $\mathcal{R}_8$ are constructed using f^+ functions and the additional contributions – to T_8 – are constructed from f^- functions.

We now consider the next-to-minimal case $n = 10$. The first line on the

right-hand side of (4.3.1) gives a non-trivial sum of S_8 contributions. These are the contributions of $\tilde{R}_8$'s and contributions of T_8's, since the latter no longer cancel each other in the sum. Novel contributions at 10-points then arise from the second line on the right-hand side of (4.3.1):

$$S_{10}(x_2, x_4, x_6, x_8, x_{10}) - S_{10}(x_1, x_3, x_5, x_7, x_9) \,. \tag{4.3.9}$$

To be cyclically symmetric in z-variables these functions have to be anti-symmetric under the flip symmetry (due to the relative minus sign in (4.3.9)). Together with T_8's these contributions from S_{10}'s will give precisely the vanishing part of the 10-point function, V_{10}[7].

We now return to the general expression (4.3.1), interestingly this formula is most simply written in terms of x-variables rather than z's. To see that this is non-trivial, imagine rewriting the right-hand side back in terms of z variables. We see that rather than having arbitrary z dependence, the z's only appear in each term as pairs of nearest neighbours, i.e. if a term depends on z_i then it will necessarily depend also on either z_{i+1} or z_{i-1}. Writing in terms of x's is a short-hand way of displaying this dependence. Furthermore, the objects S_m have properties which are similar, but nicer than the corresponding low-point amplitudes $\tilde{R}_m$. We now detail the properties of S_m for general m before proving that our formula correctly solves the constraints coming from collinear limits.

The m-point objects S_m have similar properties to the S_8-functions which we discussed above. Firstly, they are conformally invariant functions of m z-variables or equivalently $m/2$ x-variables: $S_m(z_1, \ldots z_m) = S_m(x_2, \ldots x_m)$ where $x_2 = (z_1, z_2)$ etc. They are also symmetric under cyclic symmetry in x-variables (but not necessarily in z),

$$S_m(x_2, x_4, \ldots x_m) = S_m(x_4, x_6, \ldots, x_m, x_2) \,. \tag{4.3.10}$$

and parity symmetric. Furthermore, we also require that they satisfy the additional flip (anti-)symmetry,

$$S_m(x_{i_1}, x_{i_2}, \ldots x_{i_{m/2}}) = (-1)^{m/2} S_m(x^f_{i_1}, x^f_{i_2}, \ldots x^f_{i_{m/2}}) \,. \tag{4.3.11}$$

---

[7] It was this process which was used in [47] to build one-loop N²MHV, two-loop NMHV and three-loop MHV amplitudes at 10-points.

The S_m's must also vanish in the collinear limit $z_m \to z_{m-2}$ i.e. $x_m \to x_{m-1}$

$$\lim_{x_m \to x_{m-1}} S_m(x_2, \ldots, x_{m-2}, x_m) = 0 \qquad \text{(triple collinear limits)} . \qquad (4.3.12)$$

A useful and more geometrical way of saying this would be

$$S_m(x_i, \ldots, x_j, x_k) = 0 \qquad \text{if} \qquad x_j, x_k \quad \text{become light-like separated} . \quad (4.3.13)$$

Finally S_m must also vanish in the multi-collinear limits, where $(p+1)$ consecutive momenta become collinear: $z_m, z_{m-2}, \ldots, z_{m-p+2} \to z_{m-p}$, or $x_m \to x_{m-1}, x_{m-2} \to x_{m-3}, \ldots, x_{m-p+2} \to x_{m-p+1}$ for $p = 2, 4, \ldots m - 4$ i.e.

$$S_m(x_i, x_j \ldots, x_k) = 0 \qquad \begin{array}{l} \text{if any set of } 2, 3, \ldots \text{ or } (m/2{-}2) \text{ consecutive points} \\ \text{become mutually light-like separated.} \end{array}$$

$$(4.3.14)$$

That is, we require that the S-functions vanish in *all* allowed *multi*-collinear limits. By "allowed" we mean that we cannot insist that S_m vanishes when too many points become collinear due to conformal invariance (see Sect 2.5). The limit when $m/2{-}1$ points become collinear is conformally equivalent to points being in generic positions and so S_m cannot vanish in this limit. Similarly when all $m/2$ points become collinear.

To show that (4.3.1) is indeed the n-point function, we must first prove that this expression is cyclic, that it satisfies the correct properties under collinear limits and that it is unique. That (4.3.1) is cyclically symmetric in z-variables comes straight from its definition, the (anti)-flip symmetry (4.3.11) together with cyclicity of S_m in its x-arguments. In the next subsection we argue that it behaves correctly in all collinear limits, then we discuss the uniqueness of the structure.

4.4 $\tilde{R}_n$ HAS THE CORRECT COLLINEAR LIMITS

Consider the simplest collinear limit allowable in our 2d kinematical region, the triple collinear limit $z_n \to z_{n-2}$ i.e. $x_n \to x_{n-1}$. Using

$$\lim_{x_n \to x_{n-1}} S_m(i, j \ldots k) = S_m(i, j, \ldots k) \quad \text{for} \quad i, j, \ldots k \neq n - 1, n \quad \text{and}$$

$$\lim_{x_n \to x_{n-1}} [S_m(i, j \ldots k, n{-}1) - S_m(i, j, \ldots k, n)] = 0 , \qquad (4.4.1)$$

as such, one can very easily see that we have the required result:

$$\lim_{x_n \to x_{n-1}} \tilde{R}_n(z_1, \ldots, z_n) = \tilde{R}_{n-2}(z_1, \ldots, z_{n-2}) \qquad (4.4.2)$$

as required under collinear limits.

To prove the correct property under multi-collinear limits we need to work a little harder. The multi-collinear limit, where $p+1$ edges become collinear is defined for even p as $z_n, z_{n-2}, \ldots, z_{n-p+2} \to z_{n-p}$. This is the same as pairwise limits on consecutive x-variables, $x_n \to x_{n-1}$, $x_{n-2} \to x_{n-3}$, $\ldots$, $x_{n-p+2} \to x_{n-p+1}$ as can be easily seen from Fig.2 in Chapter 2. More geometrically, we can separate all the x-variables into two sets:

$$\overbrace{x_{n-p}, x_{n-p+1} \leftarrow x_{n-p+2}, \ldots, x_{n-1} \leftarrow x_n, x_1,}^{\mathcal{S}_{p+2}} \overbrace{x_2, \ldots, x_{n-p-1}}^{\mathcal{S}_{n-p-2}} \qquad (4.4.3)$$

In this limit all points in the set $\mathcal{S}_{p+2} = \{x_{n-p}, x_{n-p+1} \ldots, x_1\}$ are becoming mutually light-like separated (i.e. collinear), whereas the points in the set $\mathcal{S}_{n-p-2} = \{x_2 \ldots, x_{n-p-1}\}$ remain unchanged. Now the S's vanish whenever r consecutive points become light-like separated for $r = 2, 3, \ldots \frac{m}{2} - 2$ as discussed in (4.3.14). Since all the points in $\mathcal{S}_{p+2}$ become light-like separated from each other, this means that S_m vanishes unless all, or all but one of the points are in $\mathcal{S}_{p+2}$ or $\mathcal{S}_{n-p-2}$, i.e.

$$S_m(i_1, \ldots i_r, j_1, \ldots j_{\bar{m}-r}) \to 0 \ \text{ for } r = 2, \ldots, \bar{m}-2$$

$$\text{and where } \{i_1, \ldots, i_r\} \in \mathcal{S}_{m-p-2} \text{ and } \{j_1, \ldots, j_{\bar{m}-r}\} \in \mathcal{S}_{p+2} , \quad (4.4.4)$$

where we have defined $\bar{m} = m/2$. So the sum of S's appearing in $\mathcal{R}$ reduces to

$$\sum_{2 \leq i_1 \triangleleft \cdots \triangleleft i_{\bar{m}} \leq n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1 + \cdots + i_{\bar{m}}}$$

$$\longrightarrow \sum_{2 \leq i_1 \triangleleft \cdots \triangleleft i_{\bar{m}-1} \leq n-p-1} \sum_{j=i_{\bar{m}-1}+2}^{n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}-1}}, x_j)(-1)^{i_1 + \cdots i_{\bar{m}-1}}(-1)^j$$

$$+ \sum_{n-p \leq j_1 \triangleleft \cdots \triangleleft j_{\bar{m}-1} \leq n+1} \sum_{i=2}^{j_1-2} S_m(x_i, x_{j_1}, x_{j_2}, \ldots, x_{j_{\bar{m}-1}})(-1)^{j_1 + \cdots j_{\bar{m}-1}}(-1)^i .$$

$$(4.4.5)$$

Now consider the first term of this last expression, and in particular focus on the sum over j. We have that

$$\sum_{j=i_{\bar{m}-1}+2}^{n+1} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}-1}}, x_j)(-1)^j$$

$$= \sum_{i_{\bar{m}}=i_{\bar{m}-1}+2}^{n-p-1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_{\bar{m}}} + \sum_{j=n-p}^{n+1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}-1}}, x_j)(-1)^j$$

$$= \sum_{i_{\bar{m}}=i_{\bar{m}-1}+2}^{n-p-1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_{\bar{m}}} + S_m(x_{i_1}, \ldots, x_{i_{\bar{m}-1}}, x_{n-p})$$

$$- S_m(x_{i_1}, \ldots, x_{i_{\bar{m}-1}}, x_{n+1}) \qquad (4.4.6)$$

where in the last equality we have used the fact that x_j is one of the vertices becoming collinear, in the limit $x_n \to x_{n-1}$, $x_{n-2} \to x_{n-3}$, $\ldots$, $x_{n-p+2} \to x_{n-p+1}$, thus the alternating sum collapses to the two boundary cases. Inserting this back into (4.4.5) and using cyclicity, we can include this most succinctly by including the end-points $n-p$ and $n+1 = 1$ in the sum to rewrite the first term on the right-hand side of (4.4.5) in the suggestive form

$$\sum_{1 \leq i_1 \lhd \cdots \lhd i_{\bar{m}} \leq n-p} S_m(x_{i_1}, x_{i_2}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1 + \cdots + i_{\bar{m}}} . \qquad (4.4.7)$$

So, we have massaged the first term on the right-hand side of (4.4.5) into a nice form. The second term in (4.4.5), despite its superficial similarity to the first term, looks decidedly trickier to manipulate into something pleasant, since instead of one-point becoming collinear $m/2-1$ of the points are becoming collinear. However, here we can make use of the fact (used in [3]) that the collinear limit we are performing is conformally equivalent to a different multi-collinear limit. In the conformally equivalent case, instead of the points in $\mathcal{S}_{p+2}$ becoming collinear and the points in $\mathcal{S}_{n-p-2}$ remaining unchanged, we have the converse: the points in $\mathcal{S}_{p+2}$ remain unchanged and the points in $\mathcal{S}_{n-p-2}$ become collinear. With this observation we see that in this conformally equivalent setting, only the point x_i is becoming collinear and the points x_j remain unchanged. We can then perform analogous manipulations to those leading to (4.4.7) on the second term on the right-hand side of (4.4.5) to obtain

the final result

$$
\sum_{2 \leq i_1 \triangleleft \cdots \triangleleft i_{\bar{m}} \leq n+1} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1 + \cdots + i_{\bar{m}}}
$$

$$
\rightarrow \sum_{1 \leq i_1 \triangleleft \cdots \triangleleft i_{\bar{m}} \leq n-p} S_m(x_{i_1}, \ldots, x_{i_{\bar{m}}})(-1)^{i_1 + \cdots + i_{\bar{m}}}
$$

$$
+ \sum_{n-p-1 \leq j_1 \triangleleft \cdots \triangleleft j_{\bar{m}} \leq n+2} S_m(x_{j_1}, \ldots, x_{j_{\bar{m}}})(-1)^{j_1 + \cdots + j_{\bar{m}}} . \qquad (4.4.8)
$$

Now this is true for any value of m and since our general formula for the amplitude (4.3.1) is made from such structures as these, we conclude that in the multi-collinear limit

$$
\tilde{R}_n \rightarrow \tilde{R}_{n-p} + \tilde{R}_{p+4} , \qquad (4.4.9)
$$

precisely as we require.

We here note that this simple analysis and presentation of collinear-limits is worthy of consideration in of itself. The technique of building amplitudes from the perspective of collinear limits is only a practicable possibility because of the neat presentation we have made of the collinear limits both in [80] and again here (4.4.9). In particular it is worth mentioning that much of the analysis for collinear limits in 2d-kinematics can be derived in an analogous manner for the full four-dimensional model and we would expect a similar pattern to emerge for the splitting amplitudes. Additionally, we expect that if a large enough number of points become collinear, then by conformal symmetry arguments, this will be equivalent to a conformal transformation and as such leave the amplitude invariant. We will make no efforts here to formalise these arguments for the full four-dimensional picture and simply leave it as an aside.

4.5   Uniqueness of the Uplift

We have just demonstrated that (4.3.1) gives a solution of the constraints from collinear limits, but how unique is this solution? To examine this question, imagine that the formula (4.3.1) failed to give the correct result for $\tilde{R}_n$ at n-points (but succeeded below this point). Then consider the difference between the prediction from (4.3.1) and $\tilde{R}_n$, $\tilde{R}_n - \tilde{R}_n^{(4.3.1)}$. Since both obey the same collinear limits, this is an n-point function which vanishes in all allowed collinear limits. So we would expect to be able to absorb this into the definition of the collinear vanishing object S_n. But at n-points the only obvious

properties $\tilde{R}_n - \tilde{R}_n^{(4.3.1)}$ must satisfy are conformal invariance, cyclic (and parity) symmetry and that it should vanish in all allowed collinear limits. Our functions S_m on the other hand possess two additional restrictions, namely:

1. $S_m^{(\ell)} = 0$ for $m > 4\ell$,

2. $S_m(x) = (-1)^{m/2} S_m(x^f)$ (flip (anti)-symmetry) .

So we need only focus on the question of why the collinear vanishing part of $\tilde{R}_n$ should possess these additional properties.

The first point, that there is no collinear vanishing, cyclically symmetric, ℓ-loop function beyond 4ℓ-points, was argued in [91]. For completeness we briefly repeat the argument, based on examining the symbol of S_m. The central assumption of [91] was that the basis of the symbol (in 2d kinematics) is made out of simple cross-ratios u_{ij}. These cross-ratios have a clear and simple behaviour in two collinear limits, those associated with the edges i or j, specifically $\log u_{ij} \to 0$ when either $z_{i+1} \to z_{i-1}$, or $z_{j+1} \to z_{j-1}$. Thus, the presence of u_{ij} in the symbol of S_m makes it vanish in the collinear limits associated with the edges i or j. To make sure that S_m vanishes in all possible collinear limits its symbol must contain u_{ij}'s for all pairs of edges. If we add terms in the symbol which are linear combination of u's they cannot impose more than 2 collinear limits, so this argument remains unchanged even after a relaxation of this assumption. At ℓ-loops, there are 2ℓ tensor products of u_{ij}'s in the symbol, and they can connect maximally 4ℓ different edges. This means that collinearly vanishing functions exist only up to $m = 4\ell$ points.

Moving on to the second point, for functions $S_{4m'}$ satisfying the flip (anti)-symmetry, we have found the unique collinear uplift, and this is valid *without any further constraints on S_m*, in particular it need not satisfy any linear identities and the formula manifestly satisfies the correct collinear behaviour. There is no such manifest collinear uplift for an $S_{4m'}$ which does not satisfy the flip symmetry as the amplitude would lose cyclicity in z-variables. To put it another way, any further solutions for these cases would have to satisfy very special non-trivial linear identities between functions defined at different points.

In later subsections we will extend this analysis to non-MHV amplitudes and obtain similar conclusions and we will also explain how the tree-level NMHV amplitudes evade these constraints precisely by satisfying such non-trivial identities. They evade our conclusions by not manifestly having the correct collinear behaviour (and indeed they are not manifestly cyclic either).

They only have these properties after taking into account linear identities which we believe to be special to tree-level, meaning that at loop level the only solution is of the form (4.3.1).

4.6 Special cases

We first return to once again considering the n-point, 2-loop result of [89]. At 2-loops, by inserting the 8-point result

$$S_8^{(2)}(z_1, \ldots, z_8) = -\frac{1}{4} \log(u_{17;53}) \log(u_{31;75}) \log(u_{28;64}) \log(u_{42;86}) \qquad (4.6.1)$$

into our MHV uplifting formula (4.3.1)

$$\tilde{R}_n^{(2)}(z_1, z_2, \ldots, z_n) = \sum_{1 \leq i_1 \lhd i_2 \lhd i_3 \lhd i_4 \leq n} S_8^{(2)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1 + \cdots + i_4} \qquad (4.6.2)$$

and finally rewriting in terms of the basis $u_{i,j}$'s, we correctly reproduce the form of the two-loop result in earlier chapters (2.9.21).

The 3-loop MHV formula (4.3.1) for any number of points contains essentially only three independent terms. It reduces to

$$
\begin{aligned}
\tilde{R}_n^{(3)}(z_1, z_2, \ldots, z_n) = &\sum_{1 \leq i_1 \lhd i_2 \lhd i_3 \lhd i_4 \leq n} S_8^{(3)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1 + \cdots + i_4} \\
&+ \sum_{1 \leq i_1 \lhd \cdots \lhd i_5 \leq n} S_{10}^{(3)}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1 + \cdots + i_5} \\
&+ \sum_{1 \leq i_1 \lhd \cdots \lhd i_6 \leq n} S_{12}^{(3)}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1 + \cdots + i_6} \,, \qquad (4.6.3)
\end{aligned}
$$

where the multi-collinearly vanishing function S_{12} is constructable using the methods of [91] as was demonstrated in [80] and later updated to include symbol entries of the form $u_1 - u_2$ [47]. In particular these methods involved writing down all possible terms allowed under cyclicity, parity, integrability of the symbol and disappearing in all twelve triple-collinear limits. We will show that the general formula correctly reproduces the 10-point result and gives the entire collinear vanishing term at 10-points, V_{10} in (4.2.14), with terms not included in [80] which contain symbol entries of the form $u_{1,5} - u_{2,6}$ being added from the appendices of [47].

S_{10} contribution to $\mathcal{R}_{10}$

We first consider the S_{10} collinear vanishing contribution to $\mathcal{R}_{10}$ constructed explicitly in the most general fashion under the assumption that only cross-ratios can occur in the symbol. At 10-points there are 10 fundamental cross-ratios

$$u_i := u_{i,i+4}, \qquad i = 1, \ldots, 10 \tag{4.6.4}$$

which can be divided into 5 even $(u_1, u_3, \ldots, u_9)$, and 5 odd cross-ratios $(u_2, u_4, \ldots, u_{10})$. It was argued in [91] that V_{10} is assembled from functions of even cross-ratios times functions of odd cross-ratios as follows:

$$f_i(u_{\text{even}}) f_j(u_{\text{odd}}) + \text{cyclic} + \text{parity}, \tag{4.6.5}$$

where these functions f_i must themselves vanish in any collinear limit. To do this they must have weight-3 and each term must contain 3 consecutive cross-ratios of given parity, e.g. u_1, u_3, u_5. They are not difficult to find analytically [91]:

$$f_1(u_1, u_3, u_5) = \log(u_1) \log(u_3) \log(u_5)$$

$$f_2(u_1, u_3, u_5) = \log(u_3) \left(\text{Li}_2(u_1) - \text{Li}_2(1 - u_3) + \text{Li}_2(u_5) - \frac{\pi^2}{6} \right)$$

$$f_3(u_1, u_3, u_5, u_7, u_9) = \sum_{i=1,3,5,7,9} (\text{Li}_3(u_i) - \text{Li}_3(1 - u_i)) - \zeta_3 . \tag{4.6.6}$$

Here f_1 and f_2 give 5 independent functions via cyclic permutations of the arguments, whereas f_3 is cyclically symmetric giving only 1 independent function, as such we have 11 functions in total.

Let us now rewrite these functions in a basis which diagonalises the action of the cyclic group[8],

$$f_1^{(k)}(z_1, z_3, z_5, z_7, z_9) := \sum_{j=1}^{5} f_1(u_{2j}, u_{2j+2}, u_{2j+4}) e^{2\pi \mathbf{i} jk/5} \qquad k = 0, \ldots, 4$$

$$f_2^{(k)}(z_1, z_3, z_5, z_7, z_9) := \sum_{j=1}^{5} f_2(u_{2j}, u_{2j+2}, u_{2j+4}) e^{2\pi \mathbf{i} jk/5} \qquad k = 0, \ldots, 4$$

$$f_3^{(0)}(z_1, z_3, z_5, z_7, z_9) := f_3(u_2, u_4, u_6, u_8, u_{10}) \tag{4.6.7}$$

Note here that the $\mathbf{i}$ in the argument of the exponentials is not an index and is instead $\mathbf{i} = \sqrt{-1}$. These new functions lie in irreducible representations of

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[8] Here we mean the cyclic group which acts separately on the even and odd variables, so in this case it is C_5

the cyclic group, in fact they are eigenstates of the cyclic group,

$$f_a^{(k)}(z_3, z_5, z_7, z_9, z_1) = e^{2\pi i k/5} f_a^{(k)}(z_1, z_3, z_5, z_7, z_9) \qquad (4.6.8)$$

Note additionally that under parity, we find $f^{(k)} \to f^{(5-k)}$. Then by construction both V_{10} appearing in (4.1.7) and S_{10} appearing in (4.3.1) are C_5 and parity invariant combinations of these functions.

Recall that the only distinction between S_{10} and V_{10} is that S_{10} contains the additional symmetry that it must be invariant under flips of the variables. To obtain cyclic (C_{10}) invariant combinations, a function carrying cyclic representation k must multiply a function carrying cyclic representation $-k$. This is very simple to see if we consider the action of cyclic symmetry on such a combination. The first term which has a cyclic representation k picks up a factor of $\exp^{2\pi i k/5}$ whereas the second term necessarily comes with a factor $\exp^{-2\pi i k/5}$ where these two factors combine to give 1. As such there is no additional factor and simply the two functions have switched roles, now the first has gone from e.g. odd variables $\to$ even variables whereas the second has switched from being a function of even variables to odd variables. As such a combination $f_a^{(k)}(z_1, z_3, \dots) f_b^{(-k)}(z_2, z_4, \dots)$ requires a term $f_a^{(k)}(z_2, z_4, \dots) f_b^{(-k)}(z_3, z_5, \dots)$ to be cyclically invariant.

$$f_a^{(k)}(z_1, z_3, \dots) f_b^{(-k)}(z_2, z_4, \dots) + f_a^{(k)}(z_2, z_4, \dots) f_b^{(-k)}(z_3, z_5, \dots) \qquad (4.6.9)$$

If we now define all functions in the set $z_{\text{odd}} = (z_1, z_3, \dots)$ and $z_{\text{even}} = (z_2, z_4, \dots)$ then we see we need to cycle the very last function backwards by two and we pick up a factor of $e^{-2\pi i k/5}$. This implementation of cyclic symmetry is used when we shortly define our complete collinear-vanishing function (4.6.10).

Let us first construct V_{10}: the most general combination of functions which vanish in all collinear limits and with no additional flip-(anti)symmetry requirements. This is given by a linear combination of the 12 collinear vanishing contributions to the remainder function listed in Appendix A. These are now written as

$$f_a^{(k)}(z_{\text{odd}}) f_b^{(-k)}(z_{\text{even}}) + e^{-2\pi i k/5} f_b^{(-k)}(z_{\text{odd}}) f_a^{(k)}(z_{\text{even}}) \quad + a \leftrightarrow b \quad (4.6.10)$$

$$f_3^{(0)}(z_{\text{odd}}) f_a^{(0)}(z_{\text{even}}) + f_a^{(0)}(z_{\text{odd}}) f_3^{(0)}(z_{\text{even}}) \qquad a = 1, 2, 3 \quad (4.6.11)$$

where $z_{\text{odd}} := z_1, z_3, z_5, z_7, z_9$ and $z_{\text{even}} := z_2, z_4, z_6, z_8, z_{10}$. In the first equation we have $a, b = 1, 2$ and $k = 0, 1, 2$ thus it gives 9-independent functions, in the

second equation $a = 1, 2, 3$ giving 3 more. Clearly these 12 functions are simple recombinations of the 12 functions in (4.6.6). We can see (4.6.10) is now cyclically invariant under C_{10}, where as we discussed the first line above (4.6.9) and the second line is constructed from functions which are themselves cyclically symmetric (4.6.6) provided they are in a combination where we can swap even↔odd variables, but we will pick up no additional factors. This is our final result for V_{10} under our assumption from [80] that all entries are simple cross-ratios.

Let us now compare this with the construction of S_{10}. These are constructed from the same building block functions, with an additional constraint that they must be antisymmetric under flip symmetry. They are given as

$$
\begin{aligned}
S_{10}(x_2, x_4, x_6, x_8, x_{10}) \ni \quad & f_a^{(k)}(z_{\text{odd}})f_b^{(-k)}(z_{\text{even}}) - f_b^{(-k)}(z_{\text{odd}})f_a^{(k)}(z_{\text{even}}) \\
& + f_b^{(k)}(z_{\text{odd}})f_a^{(-k)}(z_{\text{even}}) - f_a^{(-k)}(z_{\text{odd}})f_b^{(k)}(z_{\text{even}})
\end{aligned}
$$
(4.6.12)

Non-vanishing contributions arise from $k = 1, 2$ and $a, b = 1, 2$ so we have 6 contributions in total. Note in particular that the invariant representation $k = 0$ drops out here. Now, the contribution from S_{10}'s to $\mathcal{R}_{10}$ dictated by the S-formula (4.3.1):

$$
\tilde{R} \ni S_{10}(x_2, x_4, x_6, x_8, x_{10}) - S_{10}(x_1, x_3, x_5, x_7, x_9)
$$
(4.6.13)

becomes

$$
(1 - e^{2\pi i k/5}) \left(f_a^{(k)}(z_{\text{odd}})f_b^{(-k)}(z_{\text{even}}) + e^{-2\pi i k/5} f_b^{(-k)}(z_{\text{odd}})f_a^{(k)}(z_{\text{even}}) + a \leftrightarrow b \right)
$$
(4.6.14)

We can now see that the contribution of S_{10}'s to the 10-point amplitude (4.6.10) gives a clearly identifiable subset of the most general collinearly vanishing contribution V_{10}. They are the same functions, simply multiplied by a constant factor $(1 - e^{2\pi i k/5})$ which plays no role except in the case $k = 0$ where it vanishes.

Thus we clearly see that the contribution of S_{10} yields the entire collinear vanishing part of $\mathcal{R}_{10}$ except the pieces constructed from the cyclically invariant functions $f_a^{(0)}$. We will now see how these missing building blocks are correctly filled in by contributions from S_8 or more precisely T_8. For more details on the functions and their combinations etc. in this construction see Appendix A.

T_8 contribution to $\mathcal{R}_{10}$

Now consider the contribution of S_8 to $\mathcal{R}_{10}$ where, following our earlier scheme (4.3.5), we split S_8 into $\mathcal{R}_8$ and T_8 parts. The role of $\mathcal{R}_8$ is completely clear, it is the 8-point amplitude and furthermore it contributes to the collinearly non-vanishing part of all higher point amplitudes. The first contribution of T_8 however arises only at 10-points where it contributes to the collinearly vanishing part of the answer. Here we wish to identify and trace through the T_8 contribution to V_{10}

From Appendix A we obtain

$$T_8(x_2, x_4, x_6, x_8) = \sum_{\sigma,\tau} b_{\sigma\tau} f_\sigma^-(u_1) f_\tau^-(u_2) \tag{4.6.15}$$

where $b_{\sigma\tau} = b_{\tau\sigma}$ and the functions f_σ^-, $\sigma = 1, 2, 3$ are listed in Appendix A. These functions f_σ^- are all weight 3. It turns out that contributions of T_8 of the form (weight 2)$\times$ (weight 4) vanish at all points. We will discuss this point further at the end of this subsection. In terms of the z-variables these functions satisfy the following property.

$$f_\sigma^-(z_3, z_5, z_7, z_1) = -f_\sigma^-(z_1, z_3, z_5, z_7) \tag{4.6.16}$$

i.e. they are invariant with an alternating sign under cyclic (C_5) symmetry.

Inserting T_8 into the S-formula

$$\begin{aligned}\tilde{R}_{10} &\in \sum_{1 \le i_1 \lhd i_2 \lhd i_3 \lhd i_4 \le 10} S_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+i_2+i_3+i_4} \\ &\in \sum_{1 \le i_1 \lhd i_2 \lhd i_3 \lhd i_4 \le 10} T_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+i_2+i_3+i_4}\end{aligned} \tag{4.6.17}$$

and performing the sum we have

$$\tilde{R}_{10} \in \sum_{\sigma\tau} b_{\sigma,\tau} F_\sigma(z_1, z_3, z_5, z_7, z_9) F_\tau(z_2, z_4, z_6, z_8, z_{10}) \tag{4.6.18}$$

where

$$F_\sigma(z_1, z_3, z_5, z_7, z_9) \tag{4.6.19}$$
$$= f_\sigma^-(z_1, z_3, z_5, z_7) - f_\sigma^-(z_1, z_3, z_5, z_9) + f_\sigma^-(z_1, z_3, z_7, z_9) - f_\sigma^-(z_1, z_5, z_7, z_9) + f_\sigma^-(z_3, z_5, z_7, z_9)$$
$$= f_\sigma^-(z_1, z_3, z_5, z_7) + f_\sigma^-(z_9, z_1, z_3, z_5) + f_\sigma^-(z_7, z_9, z_1, z_3) + f_\sigma^-(z_5, z_7, z_9, z_1) + f_\sigma^-(z_3, z_5, z_7, z_9)$$

Note that the functions F_σ, although constructed from four-point building blocks, are in fact cyclically invariant 5-point functions. Furthermore, inspection of the right-hand side of (4.6.19) shows that they also vanish in all collinear limits. Thus we see that the right-hand side of (4.6.17) corresponds precisely to $k = 0$ contributions to V_{10} using the earlier cyclic-group language. We have six contributions

$$F_1F_1 \qquad F_1F_2{+}F_2F_1 \qquad F_1F_3{+}F_3F_1 \qquad F_2F_2 \qquad F_2F_3{+}F_3F_2 \qquad F_3F_3 \tag{4.6.20}$$

and these are the six previously missing contributions in V_{10}, not accounted for by S_{10} earlier in this subsection, more details are given in Appendix A.

One obvious question is what happens if we use (weight 2) $\times$ (weight 4) functions, f^-, to construct T_8. According to the above discussion this should produce a (weight 2) $\times$ (weight 4) collinear vanishing contribution to S_{10} which we know is not present in V_{10}, giving an apparent contradiction. In reality it is easy to see that all such contributions vanish. There is a unique weight 2 function $f^-(u) = \text{Li}_2(u){-}\text{Li}_2(1{-}u)$ and when we plug it into (4.6.19) we see that the corresponding function (F_σ) vanishes. When written in terms of the symbol this identity is manifest; in terms of the polylogarithms this becomes the equation

$$\text{Li}_2(u_1){+}\text{Li}_2(u_3){+}\text{Li}_2(u_5){+}\text{Li}_2(u_7){+}\text{Li}_2(u_9){-}(u_i \leftrightarrow 1{-}u_i) = \text{constant} \tag{4.6.21}$$

Writing this in terms of u_1, u_5 via the Y-system

$$u_3 = 1{-}u_1u_5 \qquad u_7 = \frac{1{-}u_5}{1{-}u_1u_5} \qquad u_9 = \frac{1{-}u_1}{1{-}u_1u_5} \tag{4.6.22}$$

is equivalent to the famous non-trivial five-term identity for the dilogarithm (Abel's pentagon relation) first discovered by Spence in 1809. We should emphasize this result, as we believe this is potentially only the simplest example of the reproduction of known (or potentially unknown) logarithmic identities which will emerge from physical motivations in systems such as this but from more complicated amplitudes.

At higher loops such as 10-loops we will have a symbol of length 20 and if there is the thus-far-observed result that the symbol of a 2d remainder function must be divided precisely evenly between $n/2$ even cross-ratios and $n/2$-odd cross-ratios then there may indeed be a series of very highly non-trivial cancellations of all other terms not of these weights e.g. (weight 12) $\times$ (weight

8) or (weight 16)×(weight 4) in analogy to (4.6.21). This would deserve further investigation to establish whether such asymmetrically-weighted terms appear after all once you go to higher-loop orders? Or do such terms disappear through non-trivial logarithmic identities and are these identities known? Here though, we merely conclude that as mentioned earlier no (weight 2) × (weight 4) contributions survive in V_{10} while weight 3 functions have already been accounted above. We also note that the contributions involving weight 2 functions f^- also disappear from V_n at all higher n.

To summarise we have demonstrated that T_8 and S_{10} together generate all possible collinear vanishing 10-point functions. And this confirms that the S-formula, at least in this instance, does not miss anything. It has subsequently been used at N^2MHV one-loop and NMHV two-loops in [47].

Higher points

This general pattern continues in a similar way to higher points. We construct S_{2m}'s from the product of collinear vanishing building block functions of even and odd z's. We choose a basis of these which diagonalise the cyclic group and call them $f_i^{(k)}$, where k is the representation of the cyclic group C_m and 'i' labels the inequivalent functions. The S-formula gives the contribution

$$S_m = a_{ij;k} \left(f_i^{(k)}(z_{\text{odd}}) f_j^{(-k)}(z_{\text{even}}) + (-1)^m f_j^{(-k)}(z_{\text{odd}}) f_i^{(k)}(z_{\text{even}}) \right.$$
$$+ \quad \text{parity)} \qquad k = 0, 1, \dots m , \qquad\qquad (4.6.23)$$

giving the contribution to (the collinear vanishing part of) $\mathcal{R}_m$

$$a_{ij;k} \left(1 + (-1)^m e^{2\pi i k/m} \right) \left(f_i^{(k)}(z_{\text{odd}}) f_j^{(-k)}(z_{\text{even}}) \right.$$
$$+ e^{-2\pi i k/m} f_j^{(-k)}(z_{\text{odd}}) f_i^{(k)}(z_{\text{even}}) + \text{parity} \right) \qquad (4.6.24)$$

This yields all possible collinear vanishing m-point amplitudes, except those built from $k = 0$ (m odd) or $k = m/2$ (m even). In other words the S-formula omits the cyclically invariant symmetric building block functions if m is odd, or in the case where m is even it misses out the functions cyclically invariant up to a sign. However, just as earlier in this subsection, the missing contributions at $2m$-points do in fact contribute at $2m+2$ points and presumably fill the full space of available functions (although we have only checked this at the 8 to 10 point level). As such we now have a proposal for the complete uplift formula

for MHV amplitudes.

4.7   COLLINEAR UPLIFT OF n-POINT N^kMHV AMPLITUDES

The general formula for lifting MHV amplitudes to higher points immediately suggests generalisation to N^kMHV superamplitudes. To do so we will need to examine odd superspace variables in 2d and the form of the collinear limit.

As discussed earlier in Chapter 2, superamplitudes can be written in chiral superspace depending on superspace coordinates $X_i = (x_i, \theta_i^A)$ where the bosonic components x_i are given in terms of 2d lightcone coordinates. Examining the implications of the light-like condition for the θ's in 2d kinematics, we find that the condition can be solved in an analogous manner to the way we write x's in terms of z's, but for the Grassmann coordinates θ's and χ's:

$$\theta_i^{\alpha A} = \begin{cases} (\chi_{i-1}^A, \chi_i^A) \, , & i \text{ even} \\ (\chi_i^A, \chi_{i-1}^A) \, , & i \text{ odd} \end{cases} . \tag{4.7.1}$$

Indeed, comparing with the supertwistor we find that the χ's are precisely the odd supertwistor variables (2.2.2) just as the z's were the bosonic twistors.

The general formula (4.3.1) giving all n-point MHV amplitudes in terms of a finite number of collinear vanishing functions generalises immediately now to the non-MHV case. Indeed the collinear limits $z_n \rightarrow z_{n-2}$ must be accompanied by identical limits for the Grassmann coordinates $\chi_n \rightarrow \chi_{n-2}$. Here one does have to be careful about the relevant speed at which we take the limit. We here take the collinear limit in a supersymmetric way. More precisely the collinear limit can be taken as a particular superconformal transformation on the relevant vertices (the details of this limit were given in Sect 2.5).

So, precisely as for the MHV case we have collinear vanishing functions, this time of the super-coordinates $S_m(X_2, X_4, \ldots, X_m)$ which satisfy cyclicity, parity, flip symmetry and the collinear vanishing properties in all (allowed) collinear limits, so

$$S_m(X_1, X_3, \ldots X_{m-1}) = S_m(X_2, X_4, \ldots X_m) = S_m(X_m, X_{m-2}, \ldots X_2) \tag{4.7.2}$$

$$\lim_{X_m \rightarrow X_{m-1}} S_m(X_2, X_4, \ldots X_m) = 0 \, . \tag{4.7.3}$$

Or more generally S_m vanishes whenever any (allowed) number of consecutive X's become light-like separated (in the supersymmetric sense: $X_1 = (x_1, \theta_1)$ and $X_2 = (x_2, \theta_2)$ are light-like separated if $x_{12}^2 = 0$ and $\theta_{12\alpha} x_{12}^{\alpha\dot\alpha} = 0$) i.e.

$$S_m(X_i, X_j \ldots, X_k) = 0 \qquad \begin{array}{l} \text{if any set of } 2, 3, \ldots \text{ or } m/2 - 2 \text{ consecutive points} \\ \text{become mutually light-like separated.} \end{array}$$

$$(4.7.4)$$

We note that $S_m(X_1, X_3, \ldots X_{m-1})$ is a function of superspace variables, and is not the same object as the MHV function $S_m(x_1, x_3, \ldots x_{m-1})$ with purely bosonic-variables from the previous subsection. The latter however is given by the zero-th order in θ expansion of the former. As we are here discussing superamplitudes, the N^kMHV label k does not appear in $\tilde{R}_n$ and S_n in formulae below, but the N^kMHV amplitudes $\tilde{R}_{n,k}$ will arise as θ^{4k} components of $\tilde{R}_n(X)$.

The general formula for the n-point amplitude is given, in exact analogy with the MHV case, by

$$\begin{aligned}
\tilde{R}_n^{(\ell)}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n) = &\sum_{1 \le i_1 \vartriangleleft \cdots \vartriangleleft i_4 \le n} S_8^{(\ell)}(X_{i_1}, X_{i_2}, \ldots, X_{i_4})(-1)^{i_1 + \cdots + i_4} \\
&+ \sum_{1 \le i_1 \vartriangleleft \cdots \vartriangleleft i_5 \le n} S_{10}^{(\ell)}(X_{i_1}, X_{i_2}, \ldots, X_{i_5})(-1)^{i_1 + \cdots + i_5} \\
&+ \sum_{1 \le i_1 \vartriangleleft \cdots \vartriangleleft i_6 \le n} S_{12}^{(\ell)}(X_{i_1}, X_{i_2}, \ldots, X_{i_6})(-1)^{i_1 + \cdots + i_6} \\
&+ \ldots \\
&+ \sum_{1 \le i_1 \vartriangleleft \cdots \vartriangleleft i_{m_{\max}/2} \le n} S_{m_{\max}}^{(\ell)}(X_{i_1}, X_{i_2}, \ldots, X_{i_{m_{\max}/2}})(-1)^{i_1 + \cdots + i_{m_{\max}/2}}
\end{aligned}$$

$$(4.7.5)$$

This constitutes our principle result which we have built to through these initial chapters. The reason for the strong similarity with the MHV case (4.3.1) is because the collinear limits restrictions function in an identical fashion. We discussed the behaviour of the R-invairants under collinear limits in the previous chapter (3.4.5) and have discussed the bosonic functions of cross-ratios $u_{i,j}$ under collinear limits extensively before too (2.9.6). As such it should be simple to see that both parts play similar roles in enforcing functions in (4.7.5) to vanish under the required number of collinear limits. The important boundary case where we have a multi-collinear limit in which superspace points in

the set $\mathcal{S}_{p+2} = \{X_{n-p}, X_{n-p+1}, \ldots, X_1\}$ become light-like separated (in the supersymmetric sense) from all other points in $\mathcal{S}_{p+2}$ (i.e. collinear) whereas the points in the set $\mathcal{S}_{n-p-2} = \{x_2, \ldots, x_{n-p-1}\}$ remain unchanged. Importantly this limit can be described by performing a conformal transformation on the points in $\mathcal{S}_{p+2}$ (see Sect.2.5). In this limit one can see that

$$\tilde{R}_n \to \tilde{R}_{n-p} + \tilde{R}_{p+4} \ , \tag{4.7.6}$$

exactly as required. The proof follows by direct analogy to the arguments in the MHV case around (4.4.1).

Thus, the only question remaining is that of how to fix how many S's are there, i.e. what is $m_{\max}$? This will depend on the loop level ℓ and the order in χ-expansion, i.e. the value of k. Based on the MHV bound, $m_{\text{MHV}} \leq 4\ell$ and the $\overline{Q}$-equation of Ref. [46] which related N^kMHV amplitudes at ℓ-loops to N^{k-1}MHV amplitudes at $(\ell+1)$-loops, one could expect that $m_{\max} = 4(\ell+k)$, which certainly matches our results at NMHV one-loop, where $\mathcal{R}_8$ is sufficient to fix all n-point amplitudes.

4.8   TREE-LEVEL NMHV AMPLITUDE

In this section we revise our procedure of the last chapter where we reduced the known n-point tree-level NMHV superamplitudes down to 2d kinematics. This was a non-trivial procedure, since each term diverges in 2d kinematics and only certain combinations are finite.

We recap that in full 4d kinematics, the tree-level NMHV amplitude can be expressed as [55, 59]

$$\mathcal{R}_{n,1}^{tree} = \frac{1}{2} \sum_{i,j} [1, i-1, i, j-1, j] \ . \tag{4.8.1}$$

where the 5-brackets (which are totally anti-symmetric in their arguments) can be written in momentum supertwistors as [100]

$$[i, j, k, l, m] = \frac{\delta^{0|4} \left(\chi^i \langle jklm \rangle + \text{cyclic} \right)}{\langle ijkl \rangle \langle jklm \rangle \langle klmi \rangle \langle lmij \rangle \langle mijk \rangle} \ . \tag{4.8.2}$$

We reduce the NMHV tree-level amplitudes to 2d kinematics and consider the first non-trivial case: the 6-point amplitude. This is

$$\mathcal{R}_{6,1}^{\text{tree}} = \frac{1}{2}\left(\langle 13456\rangle + \langle 12356\rangle + \langle 12345\rangle\right) = \frac{1}{2}\tilde{R}(1,3,5) = \frac{1}{2}\langle 135\rangle\langle 246\rangle \; .$$

(4.8.3)

Here the first equality comes directly from the general formula (4.8.1).

Due to the large number of identities, it is not clear which is the best way of representing any amplitude at low points. However, gradually a general picture began to emerge and we obtained a simple formula for the n-point, NMHV, tree-level amplitude in 2d kinematics in terms of 3-brackets (3.3.4). The result can be written

$$\mathcal{R}_{n;1}^{\text{tree}} = \sum_{1\leq j\lhd k\leq n} \tilde{R}\left(1,j,k\right)(-1)^{1+j+k} \; .$$

(4.8.4)

which at 6-points correctly reproduces (4.8.3).

Let us then compare this NMHV tree-level result with our general result for loop level superamplitudes, given in formula (4.7.5). First of all we see that the formulae are strikingly similar with the same type of alternating sum. The tree-level formula starts at 6 points however whereas (4.7.5) starts at 8-points and for our formula we claim that collinear vanishing objects S_m can only be uplifted to higher points for $m \geq 8$ *unless they satisfy very special non-trivial linear identities.* Looking closer we see that the main difference is that in the tree-level formula (4.8.4) only two indices are summed over, the first variable remaining fixed, so above in (4.8.4) we see the variable fixed at $\mathcal{Z}_1$. This does not look cyclically invariant and indeed verification of cyclic invariance requires the implementation of non-trivial linear identities between the six-point $\tilde{R}\left(i,j,k\right)$ at different points. We can of course make it manifestly cyclically symmetric by summing cyclic terms and dividing by n to give

$$\mathcal{R}_{n;1}^{\text{tree}} = \sum_{i\lhd j\lhd k\lhd i} \frac{1}{n}\tilde{R}\left(i,j,k\right)(-1)^{j+k} \; .$$

(4.8.5)

This has a form very similar to the general S-formula (4.7.5), the difference is the appearance of the rather asymmetric looking $(-1)^{j+k}$ instead of the more symmetric $(-1)^{i+j+k}$ one might expect. Indeed, imagine extending the

S-formula to $m = 6$ to give

$$\sum_{1 \leq i \triangleleft j \triangleleft k \leq n} S_6(i,j,k)(-1)^{i+j+k} \tag{4.8.6}$$

with a $(-1)^{i+j+k}$ factor. The problem is the entire result simply vanishes in this case (e.g. six-points would yield $S_6(2,4,6) - S_6(1,3,5) = 0$).

So the question remains, how does the tree-level NMHV formula get round this obstacle? The answer is that the S-formula is derived to obey *manifest* cyclicity and *manifest* collinear limits. The NMHV tree-level formula does not satisfy these requirements, but instead only satisfies cyclicity *after taking into account non-trivial linear identities.*

For example, first consider taking the triple/soft collinear limit $\mathcal{Z}_n \to \mathcal{Z}_{n-2}$ (i.e. $X_n \to X_{n-1}$) on the tree-level NMHV expression (4.8.4). This gives

$$\sum_{4 \leq j \triangleleft k \leq n} \tfrac{1}{2} \tilde{R}(2,j,k)(-1)^{j+k} \xrightarrow[X_n \to X_{n-1}]{} \sum_{4 \leq j \triangleleft k \leq n-2} \frac{1}{2} \tilde{R}(2,j,k)(-1)^{j+k}$$

$$+ \frac{1}{2} \tilde{R}(2,n-2,n) \tag{4.8.7}$$

correctly reproducing the collinear limit $\mathcal{R}_{n;1} \to \mathcal{R}_{n-2;1} + \mathcal{R}_{6;1}$ manifestly. On the other hand if we instead perform the limit $\mathcal{Z}_{n-1} \to \mathcal{Z}_{n-3}$ (i.e. $X_{n-1} \to X_{n-2}$) on the tree-level NMHV expression (4.8.4) we get

$$\sum_{4 \leq j \triangleleft k \leq n} \frac{1}{2} \tilde{R}(2,j,k)(-1)^{j+k} \xrightarrow[X_{n-1} \to X_{n-2}]{} \sum_{\substack{4 \leq j \triangleleft k \leq n \\ j,k \neq n-1,n-2}} \frac{1}{2} \tilde{R}(2,j,k)(-1)^{j+k} \tag{4.8.8}$$

$$+ \frac{1}{2} \left(\tilde{R}(2,n-3,n-1) - \tilde{R}(2,n-3,n) + \tilde{R}(2,n-2,n) \right)$$

This also correctly reproduces the collinear limit $\mathcal{R}_{n;1} \to \mathcal{R}_{n-2;1} + \mathcal{R}_{6;1}$ but only after taking into account the linear identity

$$\tilde{R}(2,n-3,n-1) - \tilde{R}(2,n-3,n) + \tilde{R}(2,n-2,n) = \tilde{R}(1,n-3,n-1) . \tag{4.8.9}$$

As such it is these non-trivial linear identities which allow NMHV tree-level amplitudes to circumvent the uplifting formula. However we expect these special conditions to only be present in the tree-level description and not beyond.

4.9   Subsequent Work on Amplitudes in Reduced Kinematics

Since the publication of [80] and the general uplift formulae for amplitudes in two-dimensional kinematics, there has been progress in both the area of building amplitudes from symmetries as we have done here and other aspects of two-dimensional kinematics for amplitudes. In this section we wish to provide a quick update on the principal results of the new work, concentrating in particular on [47] as it provided a natural next step to our work as presented above as well as some minor amendments by relaxing certain assumptions which proved too strong.

In [47] the authors used the techniques which we employed: cyclicity, parity, collinear-limits, OPE constraints and additionally a much more prominent use of the $\bar{Q}$-equation to write down the next most complicated amplitudes. Having split the SU(4) R-symmetry to SU(2)$\times$SU(2) they calculated one-loop N^{2}MHV amplitudes at 8- and 10-points, two-loop NMHV amplitudes at 8- and 10-points and three-loop MHV amplitudes at 8-points. We will neither reproduce their results here or explain the details of these calculations, however there are some important comments to be made.

As mentioned previously in this chapter, the authors of [47] used a weaker set of assumptions about the symbol content than was previously used [80, 89, 91]. This set of additional assumptions can be captured as follows:

- For octagons in 2d kinematics at all loop-orders, only six different 'letters' can appear in the symbol: $v, w, 1 + v, 1 + w, v - w, 1 - vw$. All of these can already be found at 3-loops.

- The last entry of the symbol for MHV and NMHV octagons can only be $v, w, 1 + v, 1 + w$.

- The ℓ-loop N^{k}MHV amplitude in 2d kinematics can be obtained by collinear-uplifting the octagon, dodecagon etc. up to $4(\ell + k)$-gon. That is, by uplifting basic building blocks of the type $S_8, S_{10}, \ldots S_{4(\ell+k)}$. Furthermore, the depth of the transcendental functions entering $S_{4m}^{(\ell)}$ should be at most $\ell + 2 - m$.

Where the cross-ratios v, w are defined in a subtly different way from our earlier definitions, but to all intents and purposes fulfil the same role (see Appendix A). Note that in the first term we have included the additional terms $u - v$

and $1 - uv$ which are not simply cross-ratios. We discussed the role of these in the case of the remainder function for the MHV 2-loop amplitude Sect.2.9. However from the point of the 2-loop NMHV and 3-loop MHV remainder function and above these terms can no longer be ignored as they do contribute and cannot be expressed in the symbol as simply cross-ratios. These additional terms make working beyond these amplitudes to higher complexity a calculationally difficult task, however as already explained there are still limits to where such terms can appear see Sect.2.9. Although these assumptions are all well-motivated there is still no definitive proof of these properties, we however briefly highlight work by [81, 85, 107] which attempts to give the complete dictionary for the symbol first in the full four-dimensional case and later in 2d kinematics [107].

The alteration in possible letters in the three-loop MHV case amended work done in [80, 91] by adding extra terms. We have noted this at key points during this chapter and presented the euqations for $S_8^{(3)}$ given in [47] in Appendix A alongside the earlier formulae from [80]. However, primarily this work remains unchanged and in particular the additional symbol choices do not affect the derivation of the uplifting formula.

Before we finally depart from the work presented here we wish to make a few comments about potential future directions for research and pitfalls. As observed in [47] there is an interesting resemblance between the structure of multi-loop amplitudes in $R^{1,1}$ (i.e. in $1 + 1$-dimensions) and that of correlation functions. As an example, multi-loop integrals for four-point correlation functions receive contributions from "mixing" terms e.g. $\bar{x}-x$ and $1-x\bar{x}$ at 3- and 4-loops, where x and $\bar{x}$ are related to the two cross-ratios [54]. This similarity deserves further investigation and, whilst not focussing on this, we will be looking at the correlation function $\leftrightarrow$ amplitude duality in the next chapter.

As mentioned already it would be extremely useful to find a rewriting of the $\bar{Q}$-equation which does not require an intermediate step devoid of manifest dual-conformal symmetry. This would also help in finding an easier way to impose any constraints which the $\bar{Q}$-equation places on the relations between otherwise independent symbol strings. Despite investigation (with some of this work referred to in Chapter.3) we have thus far failed to achieve such a rewriting and have only succeeded in checking that the $\bar{Q}$-equation does indeed provide restrictions independent of those from our other symmetries: cyclicity, parity, etc. Such additional restrictions would be potentially very useful in

restricting possible forms of symbol strings at higher complexity and keeping the calculations computationally feasible as we progress to higher k, ℓ and n.

It should not prove too difficult to adapt the ideas presented in this chapter to the full four-dimensional kinematics in $\mathcal{N} = 4$ since the decomposition of an amplitude into a finite number of collinearly-vanishing terms at a range of orders must still hold. Principle details about collinear-limits must be adapted and the number of terms would grow since no longer are we limited to an even number of external particles, however these alterations to the uplifting formula would not make the formula overly complicated. The barrier to the imposition of a similar scheme in the full kinematics would instead be a larger set of 'letters' in the alphabet for the symbol including some rational functions of cross-ratios, it is this last alteration which potentially would make this process infeasible.

The last obvious avenue for future research in this area is to simply attempt calculations of more complicated amplitudes in ℓ, k and n. At present the most complicated amplitudes calculated in two-dimensional kinematics are at 10-point N^2MHV one-loop, 10-point NMHV 2-loops, and 10-point MHV 3-loops amplitudes. If the 12-point amplitude was found and decomposed correctly for each of these cases then the uplifting formula (4.7.5) would solve the complete n-point amplitude in each of these cases.

We will now depart from studying two-dimensional kinematics and return to the full four-dimensional picture and the study of multi-loop amplitudes through the correlation function $\leftrightarrow$ amplitude duality. This may well have relevance for the two-dimensional kinematics as was observed in [47] however we will not draw strong connections between the two.

5

MULTI-LOOP AMPLITUDES AT 4- AND 5-POINTS FROM CORRELATION FUNCTIONS TO AMPLITUDES

In this chapter we will follow closely the work done in [6, 69] in first explaining the correlator/amplitude duality. We then use this duality to derive multi-loop amplitude results from correlation functions up to 6-loops for 4-points, and (our new results from [6]) at 5-points obtaining results for 5-loops in the parity-odd sector and 6-loops for the parity-even sector. We present many results within the chapter but leave some of the high-loop results to Appendix D.

5.1 SUMMARY OF THE CORRELATION FUNCTION $\leftrightarrow$ AMPLITUDE DUALITY

As referred to in previous chapters, planar scattering amplitudes in $\mathcal{N} = 4$ are dual to polynomial Wilson loops with lightlike edges [4, 28, 35, 57, 61]. It was also recently demonstrated that both the amplitude and the Wilson loop can be generated from n-point correlation functions of the energy-momentum tensor multiplet of the theory [2, 67, 68, 73, 74]. To this end, the operators of an n-point correlator are put on the vertices of an n-gon with lightlike edges. The relationship between correlation functions and Wilson loops is rather direct [2], since both are defined on configuration space and this connection can be made supersymmetric [1, 44, 99]. In contrast, the connection between

energy-momentum correlation functions and amplitudes is conceptually not well understood, however it does provide a fully supersymmetric integrand duality which exactly reproduces the BCFW based loop-integrands [1, 67, 68, 74]. The disc planarity of amplitudes is mapped to planarity on the sphere for the correlation functions. To be specific, the correlation functions yield the square of the amplitude integrands; here the amplitude discs are quite literally welded together akin to the hemispheres of a ball touching at the equator.

The component operators in the energy-momentum tensor multiplet are dual to supergravity states on AdS_5 by the AdS/CFT correspondence [88, 96, 108]. We use superspace to package all the component operators into one superoperator, here though we prefer to write $\mathcal{O}_\Lambda(x)$ where Λ is simply a schematic label describing the precise component in question.

Two- and three-point functions of these operators can be shown to be protected from quantum corrections. As the first non-trivial objects the four-point functions have been intensely studied at both weak-coupling in field theory perturbation theory and at strong-coupling exemplifying the AdS/CFT duality. The loop corrections to these four-point function take a factorised form [75]:

$$\langle \mathcal{O}_{\Lambda_1}\mathcal{O}_{\Lambda_2}\mathcal{O}_{\Lambda_3}\mathcal{O}_{\Lambda_4}\rangle = \langle \mathcal{O}_{\Lambda_1}\mathcal{O}_{\Lambda_2}\mathcal{O}_{\Lambda_3}\mathcal{O}_{\Lambda_4}\rangle_{\text{tree}} + \mathcal{I}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}(x_i) \times f(x_i; a) \quad (5.1.1)$$

In this equation $\mathcal{I}$ is independent of the 't Hooft coupling $a = g^2 N/(4\pi^2)$ but does depend on the particular component operators in question; all the non-trivial coupling dependence lies in the single function f. We define all the ℓ-loop integrands:

$$f(x_i; a) = \sum_{\ell=1}^{\infty} \frac{a^\ell}{\ell!} \int d^4 x_5 \ldots d^4 x_{4+\ell} f^{(\ell)}(x_1, \ldots, x_{4+\ell}) \quad (5.1.2)$$

We are aware that these f-functions may cause confusion with similarly named functions in previous chapters, however to remain in keeping with published work and other material elsewhere we will remain using the f-graph notation. It is important to note that these f-functions are distinct from f-functions from previous chapters.

The one- and two-loop contributions were computed using supergraphs [32, 71, 72, 76, 86]. In [69, 70] it was demonstrated that all the loop integrands have a surprising and hidden symmetry through permuting internal and external

variables.

$$f^{(\ell)}(x_1, \ldots \; x_{4+\ell}) = f^{(\ell)}(x_{\sigma_1}, \ldots, x_{\sigma_{4+\ell}}) \qquad \forall \sigma \in S_{4+\ell} \tag{5.1.3}$$

The $S_{4+\ell}$ invariance together with conformal covariance (these functions must conform covariantly under conformal symmetry, in practice this simply means $f^{(\ell)}$ must have conformal weight 4 at each of the $(4 + \ell)$-points, where for the integration points this will be cancelled by the d^4-integration), the absence of double-propagator terms (which follows from an OPE analysis), and planarity of the corresponding graph beyond one-loop constrains the number of undetermined parameters in an ansatz of this type severely. As such that up to three loops there is only one term in the ansatz. Indeed even at higher loops it was possible to determine $f^{(\ell)}$ up to $\ell = 6$ in combination with the criteria about the exponentiation of infra-red (IR) singularities [69, 70]. We will derive these graphs more explicitly in the next section, and much of this chapter will be devoted to their derivation and use at 4-points and 5-points.

Note that the construction of any term $f^{(\ell)}$ is in terms of squared distances x_{ij}^2 in both numerator and denominator. The graph obtained by considering the denominator factors is called an f-graph and it is these graphs we will use once we have fully explained them. These graphs provide an exceptionally compact method through which we can display the full correlator. In these diagrams we denote numerator factors by dashed lines, as such an f-graph with numerator lines provides a unique associated integrand. We will derive and explore these f-functions in greater detail in the next section, however for the moment let us simply consider an overview of their use.

Specialising temporarily to four-points, the amplitude/correlation function duality relates the four-point lightlike limit of $f(x_i; a)$ to the four-point remainder function $M_4(x_i; a)$ (i.e. the full amplitude divided by the tree amplitude) in dual-momentum space $p_i = x_i - x_{i+1}$:

$$1 + 2 \sum_{\ell \geq 0} a^\ell f_4^{(\ell)} = (M_4(x_i; a))^2 \tag{5.1.4}$$

where

$$F_4^{(\ell)}(x_1, \ldots, x_4) = (\text{external factor}) \times \lim_{x_{i,i+1}^2 \to 0} \int d^4 x_5 \ldots d^4 x_{4+\ell} \frac{f^{(\ell)}}{\ell!} \tag{5.1.5}$$

where the limits are understood mod(4) and the external factor is nothing

more than $x_{13}^2 x_{24}^2 \Pi_{1 \leq i < j \leq 4} x_{ij}^2$. We will provide a graphical interpretation of all of these objects more explicitly following the work of [69]. However as a brief summary: the limit on the left-hand side corresponds to selecting all possible 4-cycles in the f-graph and manipulating them so that each point is lightlike separated from its neighbour. Thus we split the f-graph into two disc planar amplitudes ("inside" versus "outside") and these correspond to the two planar amplitude integrands.

There has been much recent work towards an understanding of the interaction between the four-point correlation functions and their dual amplitudes [66, 69, 70]. Indeed one can use this relation in reverse to read-off the correlation function from the amplitude, and with this method $f^{(7)}$ was obtained in [106] using the corresponding 7-loop amplitude [23].

Less has been made of the fact that the very same four-point correlation function is related to particular combinations of higher-point amplitudes. This remarkable feature is simply a consequence of the fact that loop corrections of correlation functions are correlation functions with the Lagrangian inserted. However, the Lagrangian is itself an operator in the energy-momentum super-multiplet. Therefore we find that the loop-corrections of n-point correlators of energy-momentum multiplets are given by certain higher-point correlators of energy-momentum multiplets. These, in turn, are related to higher-point amplitudes via the amplitude/correlation function duality. The details of this are derived in greater detail in Appendix E and here we merely state the result:

$$\sum_{\ell \geq 0} a^\ell F_5^{(\ell)} = M_5 \overline{M}_5 \qquad (5.1.6)$$

where $F_5^{(\ell)}(x_1, \ldots, x_5)$ is constructed from the four-point correlator integrands $f^{(\ell)}$.

$$F_5^{(\ell)}(x_1, \ldots, x_5) := (\text{external factor}) \times \lim_{x_{i,i+1}^2 \to 0} \int d^4 x_6 \ldots d^4 x_{5+\ell} \frac{f^{(\ell+1)}}{\ell!} \qquad (5.1.7)$$

where here the external factor is $\frac{1}{f^{(1)}} = \Pi_{1 \leq i < j \leq 5} x_{ij}^2$, M_5 is the five-point MHV amplitude (divided by tree-level) in similarity to the four-point relations, but $\overline{M}_5$ is the NMHV amplitude not present at four-points. So to obtain the five-point amplitudes rather than the four-point amplitude we simply take the five-point lightlike limit not the four-point lightlike limit. The $f^{(n)}$ graph contributes to the 4-point amplitude at the $(n-4)$-loop level and the $(n-5)$-loop level for the five-point amplitudes.

In addition to these comments, there is still the question of how (5.1.6) uniquely determines M_5? The perturbative expansion of the right-hand side contains the parity-even part $M_5 + \overline{M_5}$ (by choosing the leading 1 in either factor) but it is also possible to obtain product terms. The (sphere) planar part of the correlator integrand, on the left-hand side of the equation, breaks into classes of terms. In analogy to earlier, taking the five-point lightlike limits corresponds to choosing a five-cycle on the graph (as opposed to a 4-cycle for the 4-point amplitude) which splits the f-graph into two disc planar pieces. The ℓ-loop integrand contains terms corresponding to a single ℓ-loop integral as well as products of m-loop and $(\ell-m)$-loop integrals. As such, equation (5.1.6) is "stratified" into an over-determined system that turns out to be beautifully consistent.

We explore the super-duality in Appendix E motivating there the principal equations given above. We use these equations firstly at four-points Sect 5.2 and discuss the precise steps required to build the correlation function from symmetries and subsequently calculate the amplitude, later we repeat these steps with five external particles (Sect.5.4). Once we have discussed the theory at four-points we will discuss work done in [69] on the four-point amplitude up to 6-loops (Sect 5.3) before later doing the same for work at five-points as we calculated in [6]: up to 6-loops in the parity-even sector and 5-loops in the parity-odd sector.

5.2   Constructing the f-functions

It is now only the f-functions which concern us, as here lies all the interesting remaining structure which we must understand to build up the amplitudes. Here we will introduce the notion of f-graphs which are drawn from the denominator factors of the f-functions (and we shall later add dotted lines indicating the numerator factors). The objects $f^{(\ell)}(x_1, \ldots, x_{4+\ell})$ are rational functions, symmetric in all $(4+\ell)$-variables, conformally covariant, with weight 4 at each point and with no double-poles. To display these functions graphically we label the vertices of a graph by the dual-space points x_i and edges denote propagators $1/x_{ij}^2$. We reproduce all f-graphs up to seven points in Fig.5.1 to see how restrictive these requirements are. Since we sum over all permutations of the vertices we need not label the graph - we sum over all possible labellings. Any vertex with degree $d>4$ must be accompanied by $d-4$ numerator lines to bring the total number of denominator lines minus numerator lines to be equal to
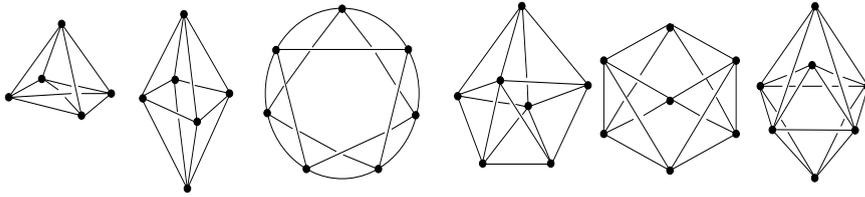
Figure 5.1: We reproduce from [69] the diagram showing all f-graphs up to 7-points (i.e. 4-point 3-loops). The five-point graph is clearly non-planar since it is K_5 (the complete graph with five vertices), however it becomes planar once we multiply it with the prefactor (E.0.6). The six-point graph $f^{(2)}$ and the second of the seven-point graphs are planar, however the remaining graphs are non-planar even after including the pre-factor in (E.0.6).
Since we know that the 4-point 3-loop amplitude is planar this will cause the contributions from the final three graphs of this figure to disappear as was shown in [69].

four. This corresponds to the fact that $f^{(\ell)}$ has conformal weight four at each external and internal point, although we sometimes suppress the numerator lines for visual simplicity (as we have done in Fig.5.1).

We now provide from [6] the f-graphs to five-loops at four-points and corresponding expressions up to three-loops, we have re-drawn them, as compared to Fig.1 in order that we can demonstrate the planarity of the higher-point amplitudes.

Recall that the graphs do not require labelling as we will sum over all possible labellings. So for example the $f^{(1)}$-graph has the equation which simply denotes the product of all propagators between all pairs of vertices, as can be seen in the graph. Whereas the $f^{(2)}$-graph is more complicated. We start with the product of all possible propagators, however we then have the wrong weighting for our points, we need to reduce the weight by one for each vertex. We do this with a sum over all possible polynomials which have the correct weight of one at each point, these will cancel with propagators in the denominator and leave us with a function of the correct weight. We need the factor of $\frac{1}{48}$ to correct for the fact that labelling the vertices in all possible ways will leave to over-counting of identical terms. After cancellation, every non-vanishing term will have the form of the $f^{(2)}$-graph. A similar story applies to the $f^{(3)}$-graph.

$$f^{(1)} = \quad \text{} \quad = \frac{1}{\prod_{1 \le i < j \le 5} x_{ij}^2} \, ,$$

$$f^{(2)} = \quad \text{} \quad = \frac{\frac{1}{48} \sum_{\sigma \in S_6} x_{\sigma_1 \sigma_2}^2 x_{\sigma_3 \sigma_4}^2 x_{\sigma_5 \sigma_6}^2}{\prod_{1 \le i < j \le 6} x_{ij}^2}$$

$$f^{(3)} = \quad \text{} \quad = \frac{\frac{1}{20} \sum_{\sigma \in S_7} x_{\sigma_1 \sigma_2}^4 x_{\sigma_3 \sigma_4}^2 x_{\sigma_4 \sigma_5}^2 x_{\sigma_5 \sigma_6}^2 x_{\sigma_6 \sigma_7}^2 x_{\sigma_7 \sigma_3}^2}{\prod_{1 \le i < j \le 7} x_{ij}^2}$$

$$f^{(4)} = \quad \text{}$$

$$f^{(5)} = \quad \text{}$$

Figure 5.2: All contributing planar f-graphs for four-points and ℓ-loops from 1-5. We have included the equations for clarity for the first few, however beyond this point it becomes increasingly lengthy to write out the equations.

We see from the above figure (which contains only the f-graphs which will ultimately contribute to our amplitudes) that $f^{(2)}$ has no need of any numerator term, whereas $f^{(3)}$ has a single numerator line which will connect the two 5-valent nodes (marked out in blue in Fig.5.2). Note the discrepancy between the number of three-loop graphs in Fig.5.2 as compared to Fig.5.1, this is due to the vanishing contribution of the non-planar graphs, see [69].

At four-points the one- and two-loop contributions were calculated using supergraphs in [32, 71, 72, 76, 86]. In contrast, three-loop results and beyond were computed using the above symmetry considerations in addition to the suppression of singularities in the coefficients [69, 70]. However, if we were to follow our earlier arguments we may consider the f-graphs of Fig.5.2 as ℓ-loop correlators at 4-points, or $(\ell - 1)$-loop correlators at 5-points etc. We will follow work done in [69] first, and consider 4-point, ℓ-loop correlators before the 5-point case is explored later (Sect 5.4).

5.3 ℓ-LOOP, 4-POINT AMPLITUDES

In planar $\mathcal{N} = 4$ SYM the duality between correlation functions and amplitudes (5.1.4) at four-points states that[9]:

$$\lim_{x_{i,i+1}^2 \to 0} \left(G_4/G_4^{(0)} \right)(x_1, x_2, x_3, x_4) = \left[(\mathcal{A}_4/\mathcal{A}_4^{(0)})(p_1, p_2, p_3, p_4) \right]^2 \qquad (5.3.1)$$

Notice that this relation (5.3.1) is one formulated in terms of integrands of the two objects not in terms of Feynman integrals. The latter diverge in the lightcone limit (for the correlation function) and for massless particles with $p_i^2 = 0$ (for the amplitude), and hence would require a regularization, say dimensional regularization to D=4-2ε dimensions (a process we will explore in detail in the next chapter). What appears on the right-hand side of (5.3.1) is the four-dimensional integrand of the amplitude, which is a rational function of the momenta. This rational function, rewritten in terms of dual coordinates ($p_i^\mu = x_i^\mu - x_{i+1}^\mu$), which we then compare with the rational integrand of the correlation function. The latter is conformally covariant by construction, while the integrand of the amplitude is known to have dual conformal invariance [16, 19, 48, 60].

We once again re-iterate that the duality (5.3.1) only applies to the planar limit of the two objects. The correlation function is known not to have non-planar corrections at one- and two-loops [32, 76] and later it was demonstrated that non-planar corrections do not appear below four-loops [69]. In contrast the four-particle amplitude begins having non-planar corrections from two-loops onwards [16, 27, 31].

The duality relation (5.3.1) involves the ratio of the correlation functions defined in the kinematical configuration in which two neighbouring operators are lightlike separated, $x_{i,i+1}^2=0$. In this limit the left-hand side becomes (5.1.4)

$$\lim_{x_{i,i+1}^2 \to 0} (G_4/G_4^{(0)}) = 1 + 2 \sum_{\ell \geq 1} a^\ell f^{(\ell)}(x_i) \qquad (5.3.2)$$

where we have relabelled for simplicity:

$$\lim_{x_{i,i+1}^2 \to 0} x_{13}^2 x_{24}^2 F_{g=0}^{(\ell)}(x_i) \to f^{(\ell)}(x_i) \qquad (5.3.3)$$

---

[9] The duality between $\mathcal{G}_n$ and $\mathcal{A}_n^{\mathrm{MHV}}$ with an arbitrary number of points was first proposed in [73, 74]. It has since been extended to the super-correlation functions of stress-tensor multiplets and non-MHV superamplitudes [1, 67, 68].

and the subscript $g = 0$ indicates that we are in the planar limit. Applying the same actions to the scattering amplitudes, the perturbative corrections to the scattering amplitudes $\mathcal{A}_4$ take the form

$$\mathcal{A}_4/\mathcal{A}_4^{(0)} = 1 + \sum_{\ell \geq 1} a^\ell \mathcal{M}^{(\ell)}(p_i) \tag{5.3.4}$$

With this the duality relation (5.3.1) reads:

$$1 + 2\sum_{\ell \geq 1} a^\ell f^{(\ell)}(x_i) = \left(1 + \sum_{\ell \geq 1} a^\ell \mathcal{M}^{(\ell)}(p_i)\right)^2 \tag{5.3.5}$$

Expanding both sides in the coupling parameter and equating coefficients (5.3.5) leads to the following series of equalities:

$$F^{(1)} = \mathcal{M}^{(1)} \qquad F^{(2)} = \mathcal{M}^2 + \tfrac{1}{2}\left(\mathcal{M}^{(1)}\right)^2 \qquad F^{(3)} = \mathcal{M}^{(1)}\mathcal{M}^{(2)} + \mathcal{M}^{(3)}$$
$$F^{(4)} = \mathcal{M}^{(1)}\mathcal{M}^{(3)} + \mathcal{M}^{(4)} + \tfrac{1}{2}(\mathcal{M}^{(2)})^2 \qquad F^{(5)} = \mathcal{M}^{(1)}\mathcal{M}^{(4)} + \mathcal{M}^{(2)}\mathcal{M}^{(3)} + \mathcal{M}^{(5)}$$
$$\tag{5.3.6}$$

and inverting these equations we find

$$\mathcal{M}^{(1)} = F^{(1)} \qquad \mathcal{M}^{(2)} = F^{(2)} - \tfrac{1}{2}\left(F^{(1)}\right)^2 \qquad \mathcal{M}^{(3)} = F^{(3)} - F^{(1)}F^{(2)} + \tfrac{1}{2}\left(F^{(1)}\right)^3$$
$$\mathcal{M}^{(4)} = F^{(4)} - F^{(1)}F^{(3)} - \tfrac{1}{2}(F^{(2)})^2 + \tfrac{3}{2}F^{(2)}(F^{(1)})^2 - \tfrac{5}{8}(F^{(1)})^4 \tag{5.3.7}$$

We use these results and our earlier discussions on how to build f-graphs from symmetry following work in [69] to give us the four-dimensional integrands of the four-particle scattering amplitude in the planar limit.

We start very simply by giving the result at one-loop, which comes from our single K_5-graph (recall that the K_5-graph is simply the complete graph on five vertices). When we combine it with prefactors etc. this gives us the simple result

$$F^{(1)}(x_1, x_2, x_3, x_4) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{(-4\pi^2)} \int d^4 x_5 f^{(1)}(x_1, \ldots, x_5) \tag{5.3.8}$$

Now $f^{(1)}$ is proportional to our K_5-graph with no kinematical dependence in the numerator.

$$f^{(1)}(x_1, \ldots, x_5) = \frac{c^{(1)}}{\prod_{1 \leq i < j \leq 5} x_{ij}^2} \tag{5.3.9}$$

In (5.3.9), the product of distances in the denominator has the required conformal weight $(+4)$ at each point, which is equivalent to the statement that the f-graph has all nodes being 4-valent. As such, the numerator has weight zero implying it is a constant. This is the only constant which is unable to be fixed without an explicit Feynman graph calculation, or alternatively comparison with known values of anomalous dimensions from OPE data. Once we substitute (5.3.9) into (5.3.8) we obtain the combination $ac^{(1)}$ in the numerator and as such we are able to absorb $c^{(1)}$ into the definition of the coupling constant. As such we set our first un-fixed constant $c^{(1)} = 1$.

The two-loop correction to the correlation function takes the form

$$F^{(2)}(x_1, x_2, x_3, x_4) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{2!(-4\pi^2)^2} \int d^4 x_5 d^4 x_6 f^{(2)}(x_1, x_2, x_3, x_4, x_5, x_6) \tag{5.3.10}$$

Recall that the function $f^{(2)}(x_1, \ldots, x_6)$ is invariant under S_6 permutations of the six-points and if we take the hexagon shown in Fig.5.2 and express the lines as propagators, we find:

$$f^{(2)}(x_1, \ldots, x_6) = \frac{c^{(2)}}{x_{13}^2 x_{14}^2 x_{15}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{25}^2 x_{26}^2 x_{35}^2 x_{36}^2 x_{45}^2 x_{46}^2} + S_6 \text{ permutations} \tag{5.3.11}$$

Again, since all vertices have degree 4 there is no kinematical dependence in the numerator, leaving only another unfixed constant $c^{(2)}$. In distinction with the five-point f-graph the four-point, two-loop graph is planar and this continues to hold for higher-loop orders as argued in Appendix A of [69].

If we place the explicit expression for $f^{(2)}$ into (5.3.10) we can then expand $F^{(2)}$ into a sum of conformally covariant scalar two-loop integrals which may appear more familiar to certain readers

$$\begin{aligned} F^{(2)} = \ & c^{(2)} \big(h(1,2;3,4) + h(3,4;1,2) + h(1,4;2,3) + h(2,3,1,4) + h(1,3;2,4) \\ & h(2,4;1,3) + \tfrac{1}{2}(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2) \left[g(1,2,3,4) \right]^2 \big) \end{aligned} \tag{5.3.12}$$

where $g(1,2,3,4)$ is the one-loop massless box-function and $h(1,2;3,4)$ is the two-loop massless ladder function:

$$\begin{aligned} g(1,2,3,4) \quad &= -\tfrac{1}{4\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \\ h(1,2;3,4) \quad &= \frac{x_{34}^2}{(4\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{x_{15}^2 x_{35}^2 x_{45}^2 x_{56}^2 x_{26}^2 x_{36}^2 x_{46}^2} \end{aligned} \tag{5.3.13}$$

Note here that all these expressions have no weight in the integration variables

and weight $+1$ in all the physical external points as required. All remaining h-integrals are obtained by permuting the indices of the external points. We find that $c^{(2)}$ is also equal to one and there are several ways (discussed in [69]) to show this, one being simply in order to reproduce the known results:

$$\mathcal{M}^{(1)} = x_{13}^2 x_{24}^2 g(1,2,3,4)$$
$$\mathcal{M}^{(2)} = x_{13}^2 x_{24}^2 \left[h(1,3;2,4) + h(2,4;1,3) \right] \tag{5.3.14}$$

noting that the term $[g(1,2,3,4)]^2$ cancels only for $c^{(2)} = 1$. Note that these relations should be understood at the level of the four-dimensional integrands. Indeed (5.3.14) matches the known one- and two-loop results [27, 31, 87].

We now, lastly, present the 3-loop correction result as it demonstrates the growth in graph topologies which has so far been hidden, and we will reserve the remainder of the four-point results to Appendix D where we give the results up to 6-loops from [69]. Let us start the 3-loop calculation from all possible f-graphs, planar and non-planar, as drawn in Fig.5.1.

$$F^{(3)}(x_1, x_2, x_3, x_4) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{3!(-4\pi^2)^3} \int d^4 x_5 d^4 x_6 d^4 x_7 f^{(3)}(x_1, \ldots, x_7) \tag{5.3.15}$$

where the S_7-symmetric function $f^{(3)}$ is given in terms of the four-topologies of Fig.5.1 as:

$$f^{(3)}(x_1, \ldots, x_7) = \sum_{\alpha=1}^{4} c_\alpha^{(3)} f_\alpha^{(3)}(x_1, \ldots x_7) = \sum_{\alpha=1}^{4} c_\alpha^{(3)} \frac{P_\alpha^{(3)}(x_1, \ldots, x_7)}{\prod_{1 \le i < j \le 7} x_{ij}^2} \tag{5.3.16}$$

Here the sum runs over $n_{\ell=3} = 4$ different P-topologies and $c_\alpha^{(3)}$ are our 4 unfixed coefficients. From our diagrams in Fig.5.1 we can read off the necessary numerator terms to be:

$$P_1^{(3)} = \frac{1}{14} x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 + S_7 \text{ permutations}$$
$$P_2^{(3)} = \frac{1}{20} (x_{12}^2)^2 (x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) + S_7 \text{ permutations}$$
$$P_3^{(3)} = \frac{1}{48} (x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) + S_7 \text{ permutations}$$
$$P_4^{(3)} = \frac{1}{48} (x_{12}^2)^2 (x_{34}^2)^2 (x_{56}^2 x_{67}^2 x_{75}^2) + S_7 \text{ permutations} \tag{5.3.17}$$

with the sum running over the S_7-permutations of the indices and the coefficients being such that each distinct term appears only once. As we have

already remarked of all the f-graphs shown in Fig.5.1 only one is planar (i.e. has genus 0) whereas the three remaining graphs have genus 1. Even after the multiplication by the prefactor on the right-hand side of (5.3.15) these graphs remain non-planar and as such might result in non-planar corrections to the correlation function $F^{(3)}$. However it was shown in [69] that this is not the case, the entire non-planar sector is ruled out by conditions on the singular behaviour of the correlation function. Instead we will have to wait until higher-points for the first non-planar contributions to appear. However we do not explore this in detail as we wish to move on to exploring this duality at 5-points and beyond. As such we refer the reader to [69] for more details at 4-points or see Appendix B, Appendix C and Appendix D for the higher-loop integrands.

5.4 Refined Duality at 5-points

Let us proceed to the more complex case of 5-point amplitudes where we will take the ρ_5^4-component of a five-point correlation function. As such (E.0.3) (the 5-point analog to (5.3.1)) will read:

$$G_{5;1}^{(\ell)} \mid_{\rho_5^4} = \frac{a^{(\ell)}}{\ell!} \prod_{i=6}^{5+\ell} \left(\int d^4 x_i \right) G_{5;1}^{(0)} \mid_{\rho_5^4} \frac{f^{(\ell+1)}(x_1, \ldots, x_{5+\ell})}{f^{(1)}(x_1, \ldots, x_5)} \tag{5.4.1}$$

At 5-points there are new allowed helicity-configurations from our earlier 4-point discussion, namely we now have NMHV amplitudes where at 5-points NMHV$\equiv \overline{\text{MHV}}$. Therefore

$$M_{5;1} = R_{1,2,3,4,5} \overline{M}_{5;0} \tag{5.4.2}$$

where $R_{1,2,3,4,5}$ is the 5-point R-invariant familiar from earlier chapters. Since there is only one independent object we will henceforth drop the second subscript on $M_{5;0}$ and write M_5 instead. Furthermore, once we take the pentagon light-like limit we will have

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{5;1}^{\text{tree}}}{G_{5;0}^{\text{tree}}} = 2R_{1,2,3,4,5} \tag{5.4.3}$$

as has been shown in [68]. The correlator/amplitude duality then implies

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_{5;1}}{G_{5;0}^{\text{tree}}} = 2R_{1,2,3,4,5} M_5 \overline{M}_5 \tag{5.4.4}$$

So combining (5.4.1), (5.4.3) and (5.4.4) and divide by the R-invariant component $2R_{1,2,3,4,5}\big|_{\rho_5^4}$ we directly obtain the relation between $f(x_i, a)$ and the five-point amplitude as quoted earlier (5.1.7):

$$\sum_{\ell \geq 0} a^\ell F_5^{(\ell)} = M_5 \overline{M}_5 \tag{5.4.5}$$

with

$$F_5^{(\ell)} := \lim_{x_{i,i+1}^2 \to 0} \frac{f^{(\ell+1)}}{\ell! f^{(1)}} \tag{5.4.6}$$

where the limit is understood mod(5). This is now an equation involving only spacetime points and was used as the basis for all higher-point calculations we performed in [6].

We next expand out the amplitude-side of this duality (5.4.5) in terms of the loop-contributions of each part of the right-hand side:

$$F_5^{(\ell)} = \sum_{m=0}^{\ell} M_5^{(m)} \overline{M}_5^{(\ell-m)} \tag{5.4.7}$$

We can also make statements about the correlation side of this duality if we proceed in our graphical manner from before. To define $F_5^{(\ell)}$ from our f-graphs we know that we have two steps, firstly we multiply by the external factor $1/f^{(1)} = \prod_{1 \leq i < j \leq 5} x_{ij}^2$ and secondly take the light-like limit. When we multiply by the external factor $\prod_{1 \leq i < j \leq 5} x_{ij}^2$ this graphically corresponds to deleting all numerator lines between external vertices, (or adding numerator lines if no denominator line is present). Taking the light-like limit means that any choice of 5-points labelled 1,2,3,4,5 will be suppressed if they are not connected via edges [1,2],[2,3],[3,4],[4,5],[5,1] in the f-graph. As such we only consider as external points, those which are connected through a 5-cycle.

Any cycle on a planar-graph necessarily divides the graph into two pieces, for example those graphs we can embed on a sphere without crossing we put the 5-cycle on the equator and as such split the graph into northern and southern hemispheres. Alternatively an embedding of the graph on the plane, a 5-cycle splits the graph into an "inside" and an "outside" graph. Armed with this insight, we now classify terms in $F_5^{(\ell)}$ according to the number m of points

inside (or outside, whichever is smaller) the corresponding 5-cycle, as

$$F_5^{(\ell)} = \sum_{m=0}^{\lfloor \ell/2 \rfloor} F_{5;m}^{(\ell)} \tag{5.4.8}$$

The classification of terms in $F_5^{(\ell)}$ according to their graph structure ($F_{5,m}^{(\ell)}$) is illustrated in Fig.5.3. A simple way to determine the value of m for any given

f−graph with 5−cycle "inside" "outside" $_\times 1/f^{(1)}$
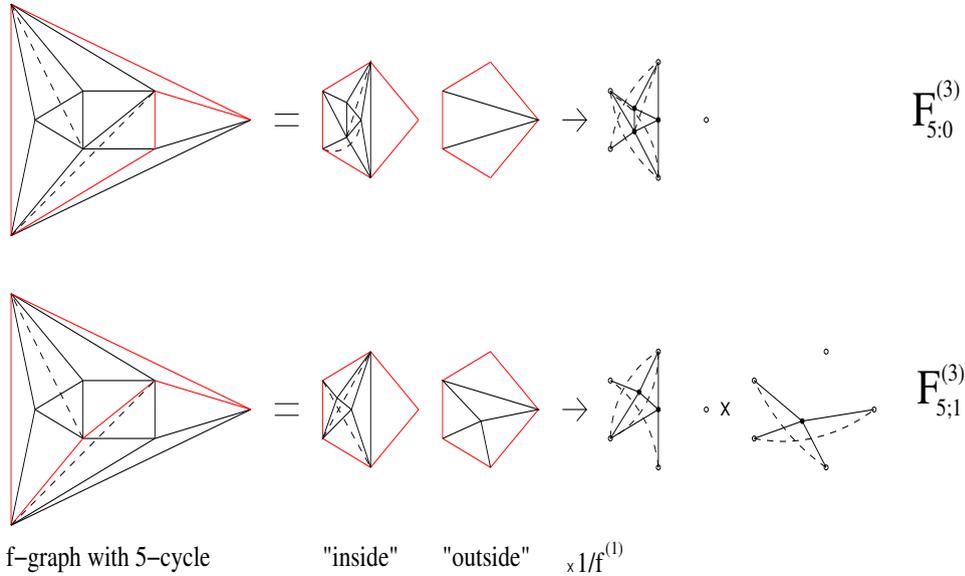
Figure 5.3: A figure as shown in [6] which illustrates the classification of $F_5^{(\ell)}$ components into classes $F_{5;m}^{(\ell)}$ starting from a single f-graph. We take two examples of 5-cycles to be our external points and these choices divide the f-graphs into two pieces, an "inside" and an "outside" with the correct planarity conditions. The minimum number of vertices inside or outside the 5-cycle gives m.

term in $F_5^{(\ell)}$ is to consider the reduced graph obtained if we delete all external vertices and edges incident to them. As such, we end up with two disconnected graphs of size m and $(\ell-m)$. In a sentence, any 5-cycle splits the f-graph into two parts and we find it useful to classify these five-cycles by the minimum number of points either inside or outside $F_{5;m}^{(\ell)}$. After the five-cycle is taken, multiplying by $\frac{1}{f^{(1)}}$ which cancels all the propagators in the five-cycle and any propagators between points on the five-cycle, alternatively any pair of points on the five-cycle which are not connected by a propagator will have an inverse propagator added in the numerator. These are the two steps demonstrated in Fig.5.3 on the same f-graph but taking different 5-cycles.

Ultimately, it should be apparent that under this process $F_5^{(\ell)}$ naturally splits into the product of two graphs precisely as the duality with the amplitude predicts $(M_5 \overline{M}_5)$. Note that this product is only imposed at the level of the denominator and we will find numerator lines connecting the two parts of the graphs. These will be considered later, however they are directly related to parity-odd terms in the amplitude.

In summary, we expect a more refined duality relating specific terms of $F_5^{(\ell)}$ to specific products of amplitudes as

$$F_{5;m}^{(\ell)} = M_5^{(m)}\overline{M}_5^{(\ell-m)} + M_5^{(\ell-m)}\overline{M}_5^{(m)} \qquad m = 0, \ldots, \lfloor (\ell-1)/2 \rfloor$$
$$F_{5;m}^{(\ell/2)} = M_5^{(\ell/2)}\overline{M}_5^{(\ell/2)} \qquad\qquad \ell \in 2\mathbb{Z} \qquad\qquad (5.4.9)$$

For this refined version of this duality to be true as stated we must be certain there can be no interaction between different terms (i.e. different values of m). The left-hand side is clearly well-defined since the inside and the outside of the 5-cycle on a planar f-graph is well defined. On the right-hand side we need to ask if all terms in $M_5^{(\ell-m)}\overline{M}_5^{(m)}$ are uniquely identified by their topology as being $(\ell-m)$-loops times m-loops objects. As such the question becomes if we draw a pentagon around $M_5^{(m)}$ say, could we also draw some or all of $M_5^{(\ell-m)}$ inside the pentagon without crossing? One can convince oneself that this is indeed not possible: $M_5^{(m)}$ contains at least four external vertices, any internal vertex of $M_5^{(\ell-m)}$ is connected to at least four external vertices and it is impossible to draw both these graphs inside a pentagon without crossing.

5.5   FOUR-POINT GRAPHS APPEAR SYMMETRICALLY

There is a simple all-loop consequence of this duality which we mention here, namely that for 5-point amplitude graphs depending on only 4 external points (i.e. with one massive external momentum), the massive point must always appear symmetrically in all four places (where allowed). This will allow us to simply express one representative of all such graphs with no ambiguity since all dihedrally-related graphs will also be present.

It emerges that four-point graphs only arise in the parity-even part of the amplitude (the general form of the parity-odd part will be discussed later, however it is sufficient to say that they will always depend on all five-points). The parity-even part of the amplitude is given by the $m = 0$ sector of $F_m^{(\ell)}$

from (5.4.9), as such

$$F^{(\ell)}_{5;\text{Parity-Even}} = M^{(\ell)}_5 + \overline{M}^{(\ell)}_5 \qquad (5.5.1)$$

The $F^{(\ell)}_m$ sector has an "inside" and an "outside" as discussed in the previous section, and for $m = 0$ the outside (say) has no vertices in it. The outside and inside must both be planar, but the inside contains a vertex which is not connected to any other point on the inside (apart from the two external points, around the pentagon) since it is supposed to be a four-point graph. As the f-graph has degree 4 or more at each point, this means there must be at least two lines attached to this point on the outside of the pentagon. The outside pentagon is then unique given planarity. In other words, the "inside" and "outside" pentagons have the following form which combines into the f-graph on the right. In this picture, the blue edges and vertex represent the four-point amplitude graph in question (with conformal weight 1 at all four points). This is essentially the process from Fig.5.3 in reverse
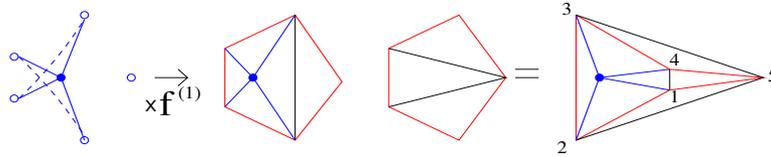
Note that in the first step multiplying by $f^{(1)}$ necessarily adds a second graph due to planarity, the two internal lines in the second graph could not be added to the first graph whilst maintaining this planarity condition. However, now we see the f-graph this four-point amplitude graph arises from, we can also see that there are a number of choices of 5-cycles all giving rise to the same amplitude graph but with the massive leg in different places:
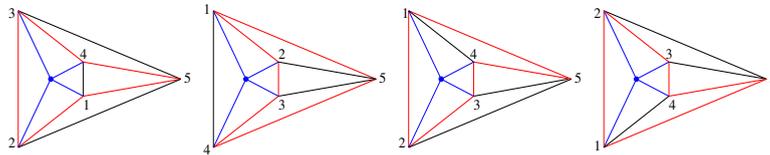
In this case the massive leg is x^2_{14} and the leg shifts its position around the amplitude. As such we can see that any four-point graph will appear symmetrically with respect to the position of its massive leg in the five-point amplitude.

The principal result from this subsection was to highlight that, for any four-point topology, the massive leg always appears in a completely symmetric manner. From this, in [6] we were permitted to display only one of our class of terms related by dihedral symmetry for reasons of brevity. We used an operator "cyc" which performed precisely this role. That is, cyc["term"] denotes the

sum over all terms related by dihedral symmetry, or swapping position of the massive leg for the four-point cases.

5.6   FIVE-POINT AMPLITUDES AT ONE- AND TWO-LOOPS

As we did for the four-point case in Sect 5.3, we now explore the one- and two-loop structures explicitly. Our duality at the lowest non-trivial order in the coupling gives

$$F_5^{(1)} = M_5^{(1)} + \overline{M}_5^{(1)} \tag{5.6.1}$$

where the correlator side $\left(F_5^{(1)}\right)$ has only one term

$$F_5^{(1)} = \text{cyc}\left[\frac{x_{13}^2 x_{24}^2}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2}\right] \tag{5.6.2}$$

where cyc[] denotes a cyclic sum of the argument, leaving us with the sum over one-mass boxes, and this is indeed the parity-even part of the one-loop amplitude.

Having found the parity-even part of the one-loop amplitude we naturally ask if we can obtain the parity-odd part too? To do this we need to go one order higher in the coupling parameter. Our refined duality (5.4.9) with $m = 1$, $l = 2$ gives

$$F_{5;1}^{(2)} = M_5^{(1)}\overline{M}_5^{(1)} \tag{5.6.3}$$

so we check that this holds. The contribution to $F_5^{(2)}$ which corresponds to product graphs (i.e. graphs with numerator lines connecting them) are given by

$$
\begin{aligned}
F_{5;1}^{(2)} &= \left(\frac{x_{13}^4 x_{24}^2 x_{25}^2}{x_{16}^2 x_{17}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{47}^2 x_{56}^2} + \text{cyclic}\,\{1,2,3,4,5\} + x_6 \leftrightarrow x_7\right)\\
&\quad + \frac{x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{35}^2 x_{67}^2}{x_{16}^2 x_{17}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2}\\
&= \text{cyc}\left[\frac{x_{13}^4 x_{24}^2 x_{25}^2}{x_{16}^2 x_{17}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{47}^2 x_{56}^2} + \frac{x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{35}^2 x_{67}^2}{x_{16}^2 x_{17}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2}\right]
\end{aligned}
\tag{5.6.4}
$$

If we equate this to $M_5^{(1)}\overline{M}_5^{(1)}$ we now have (together with (5.6.2)) two equations for our two unknowns, $M_5^{(1)}$ and $\overline{M}_5^{(1)}$. Since the equations are quadratic the

solution necessarily requires a square-root whose sign will not be determined without further information. However this sign equates to a choice of whether mostly positive helicities is MHV or alternatively NMHV at 5-points. The solution up to the sign of the square-root is simply

$$M_5^{(1)} = \frac{1}{2}\left(F_5^{(1)} \pm \sqrt{\left(F_5^{(1)}\right)^2 - 4F_{5;1}^{(2)}}\right)$$

$$\overline{M}_5^{(1)} = \frac{1}{2}\left(F_5^{(1)} \mp \sqrt{\left(F_5^{(1)}\right)^2 - 4F_{5;1}^{(2)}}\right) \tag{5.6.5}$$

Here we have written the full parity-even and parity-odd 5-point amplitudes in terms of purely parity-even objects (but with a square-root). One might legitimately ask for a better way of expressing the parity-odd piece without this square-root and indeed there is such a way.

There is a unique, parity-odd, conformally-invariant tensor, this is easiest to see in the six-dimensional formalism which we explain at the beginning of the next chapter and use extensively thereafter. In this formalism it is clear that there is a unique, parity-odd, conformally covariant object. It is a function of six-points $x_1, x_2, \ldots, x_6$ each with weight 1 which we denote ε_{123456}. It has a natural form (as suggested by our notation) in the six-dimensional formalism but can be expressed in many ways in the standard four-dimensional formalism. Through this, it can be shown that the object inside the square-root (thought of as an integrand product with integration points x_6 and x_7 which are symmetrised over) can be written in the more suggestive form

$$\left(F_5^{(1)}\right)^2 - 4F_{5;1}^{(2)} = -\frac{\varepsilon_{123456}}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}\frac{\varepsilon_{123457}}{x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2 x_{57}^2} \tag{5.6.6}$$

To see this, use the identity

$$\varepsilon_{123456}\varepsilon_{123457} =$$
$$\mathrm{cyc}\left[2x_{67}^2 x_{13}^2 x_{24}^2 x_{35}^2 x_{14}^2 x_{25}^2 + x_{13}^4 x_{24}^2 x_{25}^2 x_{46}^2 x_{57}^2 - x_{13}^4 x_{24}^4 x_{56}^2 x_{57}^2 - x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{36}^2 x_{37}^2\right] \tag{5.6.7}$$

We at last obtain our final result for the five-point amplitude integrand to be

$$M_5^{(1)} = \frac{1}{2}\left(\mathcal{I}_1^{(1)} + \mathcal{I}_2^{(1)}\right) \tag{5.6.8}$$

The terms in this amplitude are displayed graphically in Fig.5.4
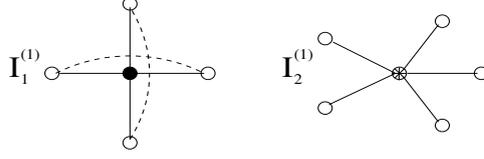
Figure 5.4: One-loop, five-point, parity-even and -odd amplitude graphs. This is simply a one-loop box in dual-coordinates, and a pentagon graph. The starred vertex v indicates a factor $i\varepsilon_{12345v}$

.

$$\mathcal{I}_1^{(1)} = \text{cyc}\left[\frac{x_{13}^2 x_{25}^2}{x_{16}^2 x_{26}^2 x_{36}^2 x_{56}^2}\right] \qquad \mathcal{I}_2^{(1)} = \text{cyc}\left[\frac{i\varepsilon_{123456}}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}\right] \qquad (5.6.9)$$

We later show that this neat and compact form of the five-point amplitude is consistent, with both the local expansion in terms of twistors [9] and with the all-orders in ε-version containing a parity-odd pentagon [26].

We now follow precisely analogous steps to those used at one-loop to derive and investigate the two-loop amplitude. The refined duality (5.4.9) gives two equations involving $M_5^{(2)}$, $\bar{M}_5^{(2)}$ and lower-loop amplitudes

$$F_{5;0}^{(2)} = M_5^{(2)} + \overline{M}_5^{(2)}$$
$$F_{5;1}^{(3)} = M_5^{(2)}\overline{M}_5^{(1)} + M_5^{(1)}\overline{M}_5^{(2)} \qquad (5.6.10)$$

Therefore, precisely as before we have two equations for two unknowns, $M_5^{(2)}$ and $\overline{M}_5^{(2)}$, from which we can solve for these quantities separately.

We first rewrite the equations as follows

$$F_{5;0}^{(2)} = M_5^{(2)} + \overline{M}_5^{(2)}$$
$$F_{5;0}^{(2)} F^{(1)} - 2F_{5;1}^{(3)} = (M_5^{(2)} - \overline{M}_5^{(2)})(M_5^{(1)} - \overline{M}_5^{(1)}) \qquad (5.6.11)$$

giving an equation for the parity-odd part of the two-loop amplitude in terms of correlator quantities and the one-loop parity-odd amplitude. Again, it is possible to simplify the parity-odd part of the amplitude at two-loops. To do this, we start with an ansatz for the form of $M_5^{(2)} - \overline{M}_5^{(2)}$. Since it is parity-odd it necessarily contains a numerator factor of the six-dimensional ε-tensor. By examination we find the parity-odd part of the two-loop amplitude

$$M_5^{(2)} - \overline{M}_5^{(2)} = \frac{1}{2!}\text{cyc}\left[\frac{\pm i\varepsilon_{123456}x_{35}^2}{x_{16}^2 x_{26}^2 x_{36}^2 x_{56}^2 x_{37}^2 x_{47}^2 x_{57}^2 x_{67}^2}\right] \qquad (5.6.12)$$

where this integrand represents a pentabox with an epsilon in the numerator. Note that the $\pm$ here is derived from the one-loop part, as such once the sign is fixed at one-loop it becomes fixed here too.

The full two-loop amplitude integrand is then

$$M_5^{(2)} = \frac{1}{2 \times 2!} \left(\mathcal{I}_1^{(2)} + \mathcal{I}_2^{(2)} + \mathcal{I}_3^{(2)} \right) \tag{5.6.13}$$

where

$$\mathcal{I}_1^{(2)} = \text{cyc} \left[\frac{x_{13}^4 x_{25}^2}{x_{16}^2 x_{17}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{56}^2 x_{67}^2} \right] \qquad \mathcal{I}_2^{(2)} = \text{cyc} \left[\frac{x_{16}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{17}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{67}^2} \right]$$

$$\mathcal{I}_3^{(2)} = \text{cyc} \left[\frac{i x_{13}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{46}^2 x_{56}^2 x_{67}^2} \right] \tag{5.6.14}$$

with the following corresponding graphs
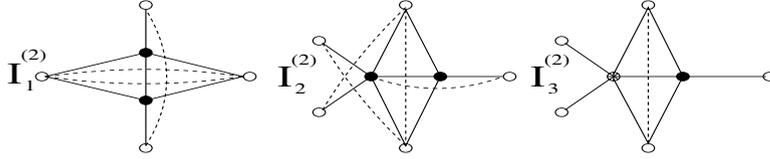
Figure 5.5: Two-loop, five-point amplitude with the parity-even $\left(\mathcal{I}_1^{(2)} \text{ and } \mathcal{I}_2^{(2)} \right)$ and parity-odd $\left(\mathcal{I}_3^{(2)} \right)$ parts shown. The starred vertex v indicates a factor $i\varepsilon_{12345v}$.

5.7 Higher Loop Amplitudes at Five-Points

It should be clear that this process can clearly be extended to higher orders. At ℓ-loops we contrive to use the refined duality (5.4.9) with $\ell, m=0$ and $\ell+1, m=1$ giving

$$F_{5;0}^{(\ell)} = M_5^{(\ell)} + \overline{M}_5^{(\ell)} \tag{5.7.1}$$

$$F_{5;1}^{(\ell+1)} = M_5^{(\ell)} \overline{M}_5^{(1)} + M_5^{(1)} \overline{M}_5^{(\ell)} \tag{5.7.2}$$

From (5.7.1) we can always immediately read-off the parity-even part $M_5^{(\ell)} + \overline{M}_5^{(\ell)}$. Then, in analogy to (5.6.11) we can write

$$\left(M_5^{(\ell)} - \overline{M}_5^{(\ell)} \right) \left(M_5^{(1)} - \overline{M}_5^{(1)} \right) = F_{5;0}^{(\ell)} F_5^{(1)} - 2 F_{5;1}^{(\ell+1)} \tag{5.7.3}$$

which will always allow us to extract the parity-odd part of the ℓ-loop graph in terms of correlator quantities (F's) and the one-loop amplitude. So knowing the right-hand side of this equation we can compute the parity-odd combination $M_5^{(\ell)} - \overline{M}_5^{(\ell)}$.

As at lower-loops we wish to rewrite this in a simpler form, i.e. in terms of ε_{123456}. In principle we could include epsilon objects with two or more internal variables, for example ε_{123467}. However we have thus far always found solutions in which only a single internal variable is present in the argument of the ε. We therefore make the following assumption.

Assumption: The parity-odd part of the five-point amplitude at any loop-order can always be written in the form $\int d^4 x_6 \ldots d^4 x_{5+\ell} \epsilon_{123456} f(x_i)$ where $f(x_i)$ is an integrand composed of x_{ij}^2 depending on all external and internal variables. There is never an epsilon-tensor involving two or more integration points.

With this assumption in place, the computation of the parity-odd contribution to the amplitude at ℓ-loops from the correlator (5.7.3) becomes remarkably straightforward. In the combination $\left(M_5^{(\ell)} - \overline{M}_5^{(\ell)} \right) \left(M_5^{(1)} - \overline{M}_5^{(1)} \right)$ on the left-hand side of (5.7.3) we consider the product of two epsilon tensors, one from ℓ-loops using the above conjecture and one from the one-loop amplitude. This product contains a single term involving an inverse propagator between two internal vertices (see (5.6.7))

$$\varepsilon_{123456}\varepsilon_{123457} = 2\mathbf{x_{67}^2} x_{13}^2 x_{35}^2 x_{25}^2 x_{24}^2 x_{14}^2 + \cdots \tag{5.7.4}$$

Thus this will produce a product graph, a pentagon around x_6 glued to a higher-loop graph involving x_7 together with a numerator x_{67}^2 between them. Such a product graph with numerator can be produced from the correlator $F_{5;1}^{(\ell+1)}$ and uniquely singles out a corresponding ε-term in $M_5^{(\ell)} - \overline{M}_5^{(\ell)}$. Similar terms will also be of interest to us in the next chapter, where two integration vertices are connected by a numerator line.

This can again be interpreted in the geometric language of correlator f-graphs: 5-cycles in the f-graph split the graph into two halves. We look for 5-cycles which have the one-loop pentagon graph on one side. The other side then gives us the parity-odd graph in question, where its coefficient is inherited from the f-graph. This procedure is illustrated in Fig.5.6. That this simple rule ultimately correctly reproduces the entirety of the parity-odd terms in (5.7.3)
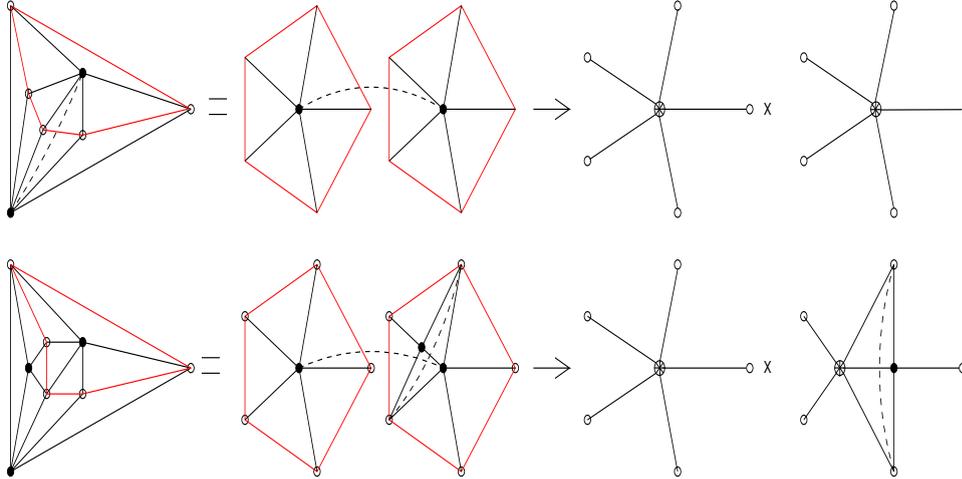
Figure 5.6: Figure illustrating the procedure for obtaining the parity-odd part of the five-point amplitude from the correlator f-graphs. The 5-cycle (shown in red) splits the graph into two parts. The inside of the 5-cycle corresponds to the one-loop parity-odd pentagon, whereas the outside corresponds to our higher-loop, parity-odd graph. The starred vertex is the vertex which is attached to the one-loop internal vertex via an internal line. In the first line we start with a 5-cycle in $f^{(3)}$ contributing to $F_1^{(2)}$, the "outside" of which determines the parity-odd graph for $M_5^{(1)}$. In the second-line we start with a 5-cycle on one of the three f-graphs in $f^{(4)}$ contributing to $F_{5;1}^{(2)}$. The "outside" of the 5-cycle then determines the parity-odd graph for $M_5^{(2)}$.

appears little less than miraculous and is reliant on many non-trivial cancellations between graphs. We attempted to give some motivation of how and why this works in the conclusion of [6] which we will review at the end of this chapter. Note that through consistency we determine many of the correlator coefficients not determined from the four-point amplitude/correlator duality (determined by the rung-rule which arises from the consistency of the four-point amplitude/correlator duality [69]). The first coefficient not determined by 5-point consistency arises in $f^{(6)}$.

There are still further consistency requirements in these equations arising from four-loops and above. Starting at four-loops, this arises since we require the $m=2$ part of $F_5^{(4)}$ to be given by the product of two two-loop amplitudes (determined through $F_{5;0}^{(2)}$ and $F_{5;1}^{(3)}$ i.e. $F_{5;2}^{(4)} = M_5^{(2)} \overline{M}_5^{(2)}$).

We used this method to obtain the full three-loop, five-point amplitude integrand (parity-even and parity-odd part) and checked that it indeed satisfies

the consistency condition (5.7.3):

$$M_5^{(3)} = \frac{1}{2}\left(\sum_{i=1}^{13} c_i \mathcal{I}_i^{(3)}\right) \tag{5.7.5}$$

where

$$c_1 = \cdots = c_6 = c_9 = \cdots = c_{12} = 1 \qquad c_7 = c_8 = c_9 = -1 \tag{5.7.6}$$

and

$$\mathcal{I}_1^{(3)} = \left(\frac{x_{13}^6 x_{25}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{28}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{57}^2 x_{67}^2 x_{68}^2}\right)$$

$$\mathcal{I}_2^{(3)} = \left(\frac{x_{16}^4 x_{24}^2 x_{25}^2 x_{35}^2}{x_{17}^2 x_{18}^2 x_{26}^2 x_{28}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_3^{(3)} = \left(\frac{x_{13}^4 x_{25}^2 x_{35}^2 x_{46}^2}{x_{16}^2 x_{18}^2 x_{28}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2}\right)$$

$$\mathcal{I}_4^{(3)} = \left(\frac{x_{13}^4 x_{24}^2 x_{46}^2}{x_{16}^2 x_{18}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_5^{(3)} = \left(\frac{x_{14}^2 x_{16}^2 x_{24}^2 x_{25}^2 x_{37}^2}{x_{17}^2 x_{18}^2 x_{26}^2 x_{28}^2 x_{36}^2 x_{46}^2 x_{47}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_6^{(3)} = \left(\frac{x_{16}^2 x_{24}^2 x_{25}^4 x_{35}^2}{x_{18}^2 x_{26}^2 x_{27}^2 x_{28}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{78}^2}\right)$$

$$\mathcal{I}_7^{(3)} = \left(\frac{x_{13}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{18}^2 x_{26}^2 x_{28}^2 x_{36}^2 x_{37}^2 x_{47}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2}\right)$$

$$\mathcal{I}_8^{(3)} = \left(\frac{x_{13}^2 x_{14}^2 x_{35}^2}{x_{16}^2 x_{17}^2 x_{36}^2 x_{38}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_9^{(3)} = \left(\frac{i x_{13}^4 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{28}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{56}^2 x_{67}^2 x_{78}^2}\right)$$

$$\mathcal{I}_{10}^{(3)} = \left(\frac{i x_{13}^2 x_{14}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{28}^2 x_{36}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{57}^2 x_{67}^2 x_{68}^2}\right)$$

$$\mathcal{I}_{11}^{(3)} = \left(\frac{i x_{24}^2 x_{36}^2 \varepsilon_{123456}}{x_{16}^2 x_{26}^2 x_{28}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_{12}^{(3)} = \left(\frac{i x_{14}^2 x_{27}^2 \varepsilon_{123456}}{x_{17}^2 x_{18}^2 x_{26}^2 x_{28}^2 x_{36}^2 x_{46}^2 x_{47}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right)$$

$$\mathcal{I}_{13}^{(3)} = \left(\frac{i x_{13}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{26}^2 x_{36}^2 x_{38}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2}\right) \tag{5.7.7}$$

also illustrated graphically in Figs.5.7 and 5.8, where each of these above terms require cycling.
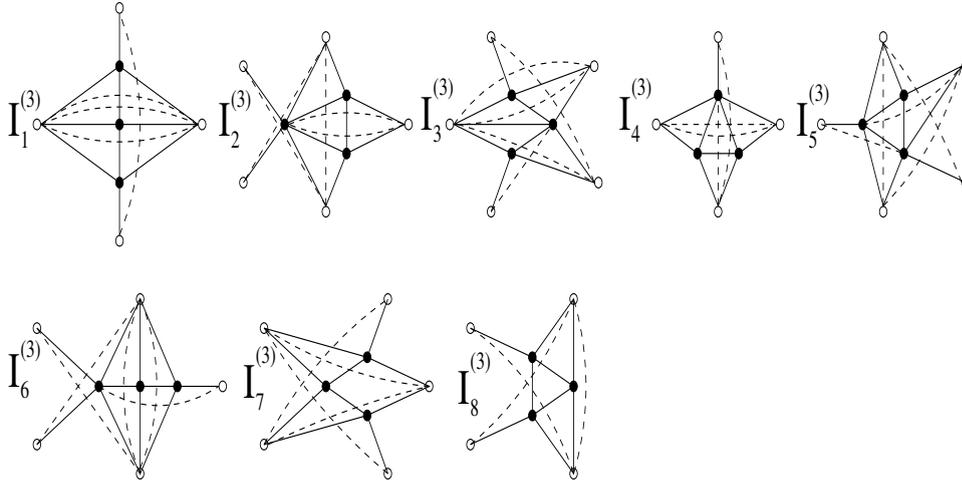
Figure 5.7: Three-loop, five-point, parity-even amplitude graphs. White vertices indicate external points x_1, x_2, x_3, x_4, x_5 which are cyclically ordered in either orientation, whereas black nodes are integration variables x_6, x_7, x_8. A black line between vertices i, j represents a propagator term $1/x_{ij}^2$. A dotted-line indicates an inverse propagator x_{ij}^2. The integrand expression $\mathcal{I}_i^{(3)}$ is the expression obtained by summing over all possible, distinct labellings.
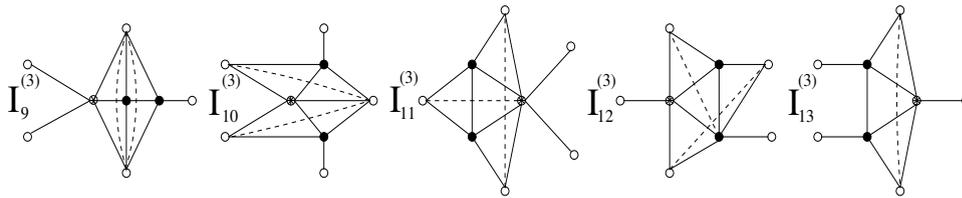
Figure 5.8: Three-loop, five-point, parity-odd amplitude graphs. A starred vertex v indicates a factor ε_{12345v}.

In Appendix.C we give the results of above analysis to obtain all four-point planar and non-planar graphs. In [6] we additionally obtained the full five-loop, parity-even and parity-odd amplitude. For this calculation we required $f^{(7)}$ which was obtained in [106] from the four-point, seven-loop amplitude [23]. The seven-loop f-graphs and their coefficients were contained in the two separate files 7LoopTopologies.txt and 7LoopCoefficients.txt attached to the arXiv version of [6]. The result for the five-point amplitude consists of 318 different parity-even topologies and 203 parity-odd graphs, which were given in the file 5pointamplitude.txt which also contained the six-loop parity-even integrand. As a piece of complementay information we included in 5pointamplitudenumberofterms.txt, the number of independent terms obtained from every graph

in 5pointamplitude.txt by the cyc [] operation. In order to obtain the parity-odd part of the six-loop amplitude we would have needed $f^{(8)}$ which could be obtained for example directly from the four-point, eight-loop amplitude were it to become available.

In the next chapter we turn to discussing alternative representations of these amplitudes allowing us to relate the results of this chapter to those written in momentum-twistors and other ways. In particular we use six-dimensional notation for which the ε-tensor becomes very natural. The aim is to discuss how we might obtain the $\mathcal{O}(\epsilon)$ terms, after we dimensionally regularise from $d = 4 \to 4-2\epsilon$ dimensions, into the scheme. However let us first discuss the results presented in this chapter, first derived in [6].

5.8   CONCLUSIONS

The supersymmetric correlator/amplitude duality in $\mathcal{N} = 4$ provides a way of relating objects with different numbers of outer points, or in- or out-going particles respectively. In [6] and again in this chapter we exploited this feature of the construction, deriving the integrand of the colour-ordered, five-point amplitude up to five (and in the parity-even sector, six-) loops, from that of the four-point function of energy-momentum multiplets. Previously this method has been chiefly associated with the MHV four-point amplitude [69, 70].

The modification required to adapt from four- to five-points is that one of the four-point integrand vertices must now be regarded as an outer-point. Necessarily we lose one loop-order form our f-graphs in this way. It turns out that the five-point integrand can only be uniquely fixed by taking into account topological information: amplitude graphs are disc planar, while the correlator integrands also contain products of two such graphs and are planar on the sphere. We used the "(one-loop)×(higher-loop)" terms to gain more equations on the loop corrections to the five-point amplitude. However, stripping off a one-loop amplitude implies losing another loop-order.

A beautiful picture emerged where the parity-even, five-point, ℓ-loop amplitudes correspond to the outside of those five-cycles in the planar correlator $f^{(\ell+1)}$-graphs which have no vertices on their insides. In contrast the parity-odd amplitude graphs correspond to the outside of those five-cycles in the planar correlator $f^{(\ell+2)}$-graphs with a single vertex on their inside, once the correct numerator replacement has been made.

Our main new results were the four- and five-loop integrands for the five-point MHV (or in this case equivalently the NMHV) amplitude. To this end, the analysis of [69] was extended to the seven-loop integrand of the four-point correlation function of energy-momentum multiplets. This in turn, was based on the result [23] for the four-point MHV amplitude up to seven-loops. As such, we modified a four-point amplitude into a five-point amplitude.

That this picture works out to be consistent is rather remarkable and non-trivial. The duality with four-point amplitudes can be shown to be consistent so long as the corresponding amplitude graphs obey the rung rule [31], (which we have not discussed here) which in the correlator picture simply corresponds to gluing pyramids onto the f-graphs [69]. Indeed, the mere existence of the four-point duality predicts many of the coefficients of loop-level amplitudes (all up to three-loops, the first two out of three four-loop f-graphs, and the first six out of seven five-loop f-graphs (see [21] etc.)). What is the topological reason stopping certain four-point, f-graphs being determined from lower loops? Recall the refined four-point duality $2F_{4;m}^{(\ell)} = M_4^{(m)} M_4^{(\ell-m)}$. Thus f-graphs with four-cycles and with a non-trivial "inside" and "outside" (i.e. which contribute to $m > 0$) are determined entirely in terms of lower-loop amplitudes. Conversely f-graphs which give no contribution to $F_m^{(\ell)}$ for $m > 0$ i.e. which have no such four-cycle, cannot be determined from lower loop four-point amplitudes (see the final two graphs of $f^{(4)}$ and $f^{(5)}$ in Fig.5.2).

For the five-point duality on the other hand the consistency is much more subtle and we have no clear understanding (i.e. a generalisation of the pyramid gluing rung rule) for why this works. The confusion comes from the many terms which appear when gluing two ε_{123456} together, many of which have to cancel. However we remarked that the structure does indeed determine many of the non-rung-rule-determined coefficients. Indeed merely the structure and consistency of the picture determines all coefficients up to $f^{(5)}$. That is, the mere existence of the amplitude/correlator duality at 4- *and* 5-points determines the four-point correlator and amplitude to five-loops and the five-point amplitude to four-loops (parity-even) and three-loops (parity-odd). The first coefficient which remains undetermined by these purely structural arguments is that of the 10-point (6-loop) f-graph:
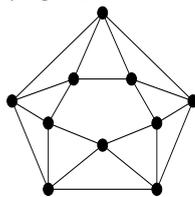
This is the first graph where it is not possible to find a five-cycle with something both inside and outside. Clearly any f-graph giving no contribution to $F_{5;m}^{(\ell)}$ for $m > 0$ (i.e. whose 5-cycles have either no vertices inside or alternatively none outside) will not be determined by lower-loops and it seems likely that the converse is true also: any f-graph contributing to $F_{5;m}^{(\ell)}$ for $m > 0$ will be determined from lower loops via the refined duality (5.4.9) $F_{5;m}^{(\ell)} = M_5^{(m)}\overline{M}_5^{(\ell-m)} + M_5^{(\ell-m)}\overline{M}_5^{(m)}$ [10]. Indeed we see that all the 5-cycles of the graph above have either nothing inside (or nothing outside) them and this is the first such f-graph, confirming this idea. Interestingly this graph is also the first f-graph with a coefficient different from ± 1, as it has the coefficient 2.

The integrands we find are given in a local form in configuration space, which is very closely related to the twistor integrands of [9, 10]. As we will demonstrate in the following chapter: the twistor numerators involving parity-odd parts can be rather painlessly rewritten in terms of simple squares of distances and the structure $\varepsilon_{12345v} = \varepsilon(X_1 X_2 X_3 X_4 X_5 X_v)$ where the X are coordinates on the projective lightcone in 6d related to those of Minkowski space (see next chapter). This object is conformally invariant and can be broken down to a sum 4d terms of the type $x_{1v}^2 \varepsilon_{x_{2v}x_{3v}x_{4v}x_{5v}}$ In the 6d epsilon 1,2,3,4,5 denote the outer points, and only the sixth variable is an integration point. All parity-odd terms in our result are of this type; epsilon terms with more than one integration vertex do not occur. By the use of Schouten identities etc. one can remove any given point from an epsilon contraction, but at the expense of introducing further denominator factors. Hence there is freedom as to the writing of the end result, although the form we found is perhaps the most natural one since it is manifestly free of higher poles like $1/x_{ij}^4$.

Interestingly, it is possible to generate the parity-even part of the five-point amplitude from the parity-odd part up to four-loops using a few universal rules for how to replace an epsilon term. These rules depend on the other numerator terms multiplying the ε_{12345v}. For example, clearly the one-loop result can be rewritten as a single pentagon: upon replacing

$$i\varepsilon_{123456} \to (x_{16}^2 x_{24}^2 x_{35}^2 + x_{26}^2 x_{14}^2 x_{35}^2 + x_{36}^2 x_{14}^2 x_{25}^2 + x_{46}^2 x_{13}^2 x_{25}^2 + x_{56}^2 x_{13}^2 x_{24}^2 + i\varepsilon_{123456})$$

(5.8.1)

---

[10] This is a little subtle since we only determine the parity-odd part of $M_5^{(\ell-1)}$ from $f^{(\ell)}$ itself. However the parity-even part also contributes to this formula, and so unless there is complete cancellation between parity-even and parity-odd which seems unlikely, $F_{5;m}^{(\ell)}$ and the corresponding f-graph will be determined by the lower-loop amplitude.

This is the only parity-odd graph with a numerator involving an ε and nothing else. Other numerators have various x^2 products multiplying the parity-odd tensor. If we make the following replacements for $a, b, c > 0$:

$$
\begin{aligned}
ix_{13}^{2a}\varepsilon_{123456} &\to x_{13}^{2a}(x_{56}^2 x_{13}^2 x_{24}^2 + x_{46}^2 x_{13}^2 x_{25}^2 + x_{26}^2 x_{14}^2 x_{35}^2 + i\varepsilon_{123456}) \\
ix_{13}^{2a}x_{14}^{2b}\varepsilon_{123456} &\to x_{13}^{2a}x_{14}^{2b}(x_{56}^2 x_{13}^2 x_{24}^2 + x_{26}^2 x_{14}^2 x_{35}^2 - x_{16}^2 x_{23}^2 x_{45}^2 + i\varepsilon_{123456}) \\
ix_{13}^{2a}x_{24}^{2b}\varepsilon_{123456} &\to x_{13}^{2a}x_{24}^{2b}(x_{56}^2 x_{13}^2 x_{24}^2 + x_{26}^2 x_{14}^2 x_{35}^2 + x_{36}^2 x_{14}^2 x_{25}^2 + i\varepsilon_{123456}) \\
ix_{13}^{2a}x_{14}^{2b}x_{24}^{2c}\varepsilon_{123456} &\to x_{13}^{2a}x_{14}^{2b}x_{24}^{2c}(x_{56}^2 x_{13}^2 x_{24}^2 + x_{26}^2 x_{14}^2 x_{35}^2 \\
&\qquad\qquad - x_{16}^2 x_{23}^2 x_{45}^2 - x_{46}^2 x_{13}^2 x_{25}^2 + i\varepsilon_{123456})
\end{aligned}
$$

$$(5.8.2)$$

and all forms related by cyclicity related, in an analogous manner, then the parity-odd graphs will give the parity-even graphs for free up to four loops. Beyond one loop, the easiest case to check is obviously the two-loop case, where we use the first replacement. This procedure fails for the first time at 5-loops where we are left with a single parity-even graph which is not determined by the parity-odd sector in this manner:
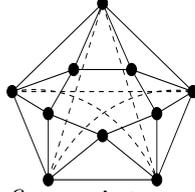
This happens to be the single five-point amplitude graph, generated by the 10-point f-graph above, whose coefficient is undetermined by consistency with the duality. So we see that these rules for obtaining parity-even graphs from parity-odd are intimately related to the consistency of the whole system but we have not fully probed this.

Note that the twistor numerators of [9, 10] also combine even and odd graphs, (we will see more of this in the next chapter) and so the above re-writing may give expressions closer to those. One direction for future work might indeed be to search for a universal numerator describing higher-loop n-point amplitudes.

Another direction for future work would be to consider the six-point light-like limit. Defining

$$
F_6^{(\ell)} := (\text{external factor}) \times \lim_{x_{i,i+1}^2 \to 0} \int d^4 x_7 \ldots d^4 x_{6+\ell} \frac{f^{(\ell+2)}}{\ell!} \tag{5.8.3}
$$

where here the limit is understood mod(6) and the external factor is $x_{12}^2 x_{13}^2 x_{15}^2$ $x_{16}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{34}^2 x_{35}^2 x_{45}^2 x_{46}^2 x_{56}^2$ then we will find the formula

$$\sum_{\ell \geq 0} a^\ell F_6^{(\ell)} = M_6 \overline{M}_6 + \text{NMHV contribution} \tag{5.8.4}$$

There are various complications which arise here. Firstly the NMHV contribution needs to be separated out (although this may be possible due to singularities in x_{14}^2, x_{25}^2 and x_{36}^2 which can only appear here and not in the MHV sector). Another complication arises since there is no longer a distinction between product graphs and disc planar graphs. The graph (one-loop box) $\times$(one-loop box) can appear in a disc planar fashion and indeed does appear in the two-loop, six-point result. Nonetheless we have seen that one can obtain more information than appears at first sight from these considerations and this certainly deserves further investigation.

6

$\mathcal{N} = 4$ SYM Planar Scattering Amplitudes to All Orders in the Dimensional Regularization Parameter

In this chapter we first introduce six-dimensional notation for our external data of planar amplitudes, we subsequently use this notation to write down the planar one-loop and two-loop MHV amplitudes at five- and six-points. We extend this analysis by dimensionally regularising the amplitudes and obtaining the contributions at $\mathcal{O}(\epsilon)$. The eventual aim of this analysis, to restrict the possible planar contributions at order epsilon and even to provide an ansatz for such terms at higher order, in a purely dual-conformally invariant basis.

6.1   4d Minkowski coordinates in 6d X-variables

One can view conformally-compactified 4-dimensional Minkowski space as a quadric inside $\mathbb{RP}^5$. Specifically we can describe Minkowski space in terms of six projective coordinates X_I living in 2+4 dimensions and satisfying the null condition

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 0 \qquad (6.1.1)$$

As such the conformal group $SO(2,4)$ acts linearly on these coordinates. The 4-dimensional Minkowski space coordinates x^μ, $\mu=0,1,2,3$ can be obtained easily from these by choosing a suitable representation for the homogeneous

coordinates X^I.

$$X^I \sim \left(\frac{1-x^2}{2}, x^\mu, \frac{1+x^2}{2}\right)^T \qquad I = -1, 0, 1, 2, 3, 4 \qquad (6.1.2)$$

Note that of course we could generalise this process of embedding our four-dimensional space into d-dimensional coordinates. It is possible to rewrite these X-coordinates in the four-dimensional spinor representation (using that SO(2,4) $\sim$ SU(2,2)), it is this representation which we employ later to consider the integrands. As such we will consider this case in detail, there are two versions:

$$X^I \to X = (\Sigma_I)X^I$$
$$X^I \to \tilde{X} = (\tilde{\Sigma}_I)X^I \qquad (6.1.3)$$

with the relation

$$\tilde{X}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}X^{\gamma\delta} \qquad (6.1.4)$$

concretely we make the following specific choice for six-dimensional Γ matrices

$$\Gamma_1 = \begin{pmatrix} 0 & \Sigma_I \\ \tilde{\Sigma}_I & 0 \end{pmatrix}$$

$$\Sigma_{-1} = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \qquad \tilde{\Sigma}_{-1} = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

$$\Sigma_\mu = \begin{pmatrix} 0 & \epsilon\sigma_\mu \\ \epsilon\tilde{\sigma}_\mu & 0 \end{pmatrix} \qquad \tilde{\Sigma}_\mu = \begin{pmatrix} 0 & -\sigma_\mu\epsilon \\ -\tilde{\sigma}_\mu\epsilon & 0 \end{pmatrix}$$

$$\Sigma_4 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \qquad \tilde{\Sigma}_4 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

where the Γ's are 8x8 matrices, the Σ's are 4x4 matrices, $\sigma_0 = \tilde{\sigma}_0 = \mathbb{I}_2$ and $\sigma^i = -\tilde{\sigma}^i$ are the standard Pauli matrices and finally

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In particular the important property which must be satisfied by this choice is

$$(\tilde{\Sigma}_I)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(\Sigma_I)^{\gamma\delta} \qquad (6.1.5)$$

giving (6.1.4), together with the Clifford algebra relations,

$$\{\Gamma_I, \Gamma_J\} = 2\eta_{IJ} \Rightarrow \Sigma_I \tilde{\Sigma}_J + \Sigma_J \tilde{\Sigma}_I = \tilde{\Sigma}_I \Sigma_J + \Sigma_I \tilde{\Sigma}_J = 2\eta_{IJ} \qquad (6.1.6)$$

where η_{IJ} is the flat metric in 2+4 dimensions. These relations imply the following for any 6-vectors X_I, X_J

$$X_{I,\alpha\beta} \tilde{X}_J^{\beta\alpha} + X_{J,\alpha\beta} \cdot \tilde{X}_I^{\beta\alpha} = 2X_I \cdot X_J \qquad\qquad X_{I,\alpha\beta} \tilde{X}_I^{\beta\alpha} = X_I \cdot X_I \qquad (6.1.7)$$

we now turn to looking at conformal covariants and in turn to how we can use this notation to express our integrands in terms of single- or double-traces etc.

Conformal Covariants

We now consider how we are to go about writing down conformal covariants. This can be done either using vector X's or spinorial X's, in the vectorial notation we use η_{IJ} or ϵ_{IJKLMN} to form invariants, those obtained using a single ϵ_{IJKLMN} will be parity-odd. The covariants for 5-points and below must necessarily be composed of only $(X_I \cdot X_J)$ (6.1.7) whereas at six-points and above we may also have the parity-odd part $\epsilon(X_1, X_2, X_3, X_4, X_5, X_6)$ etc. Indeed one can see that at six-points this is the unique parity-odd covariant piece. One can convert these invariants to four-dimensional notation straightforwardly by inputting

$$X_I \cdot X_J \rightarrow (x_i - x_j)^2 \qquad (6.1.8)$$

$$\epsilon(X_1, X_2, X_3, X_4, X_5, X_6) \rightarrow \sum_{\sigma \epsilon S_\sigma} (-1)^\sigma x_{\sigma(1)}^2 \epsilon(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})$$

$$(6.1.9)$$

As such at six-points we can construct the invariant

$$\frac{\epsilon(X_1, X_2, X_3, X_4, X_5, X_6)}{(X_1 \cdot X_4)(X_2 \cdot X_5)(X_3 \cdot X_6)} = \sqrt{\Delta} \qquad (6.1.10)$$

where $\Delta = (u_1 + u_2 + u_3 - 1)^2 - 4u_1 u_2 u_3$ is the combination appearing in the 6-point 2-loop remainder function [77, 78].

Equally we could consider covariants in the spinor representation, these can be obtained simply by taking traces of the matrices $X, \tilde{X}$ multiplied together e.g. $\text{Tr}(\tilde{X}_1 X_2 \ldots \tilde{X}_{2k-1} X_{2k})$. Using (6.1.2) we can convert these into

4-dimensional x-notation, the ϵ terms cancel and the other terms all go together to form differences of x's

$$\mathrm{Tr}(\tilde{X}_1 X_2 \ldots \tilde{X}_{2k-1} X_{2k}) \to \mathrm{Tr}(\tilde{x}_{12} x_{23} \ldots \tilde{x}_{2k-1,2k} x_{2k,1}) \tag{6.1.11}$$

Momentum Twistor Integrands in X-space

Another advantage of the X's in their spinorial notation is their direct relationship with momentum twistors, meaning it is straightforward to express twistor integrands in traces of X's. If X_i are lightlike separated or two copies of the same X_I (i.e. $X_i \cdot X_{i+1} = 0 \; \forall \; i$) then we define

$$X_{i,\alpha\beta} := Z_{i-1,[\alpha} Z_{i,\beta]} \tag{6.1.12}$$

We will refer to these as bi-twistors throughout although to be specific they are not yet true momentum twistors since they only satisfy lightlike separation. However in a particular co-ordinate patch the restrictions to be twistors would be met and then these would indeed be true bitwistors. While we will not go into the details of this coordinate patch it should be acceptable to the reader that to call these $X_{i,\alpha\beta}$ bi-twistors is not unreasonable.

Now let's consider various integrands as they are expressed in [9], starting with a dual-conformal pentagon integral in terms of which one can write the integrand of all one-loop amplitudes to at least $\mathcal{O}(\varepsilon)$:

$$I_{i,j} = \frac{\langle AB(i-1,i,i+1) \cap (j-1,j,j+1)\rangle \langle X,i,j\rangle}{\langle AB,i-1,i\rangle\langle AB,i,i+1\rangle\langle AB,j-1,j\rangle\langle AB,j,j+1\rangle\langle ABX\rangle} \tag{6.1.13}$$

The numerator can be trivially rewritten through permutations of the entries in each four-bracket, as such we could express it in the following way $\langle i-1,i,i+1,[A\rangle\langle B] j-1,j,j+1\rangle\langle j,X,i\rangle$ where there is an antisymmetry across $A \leftrightarrow B$. This form along with definitions (6.1.4), (6.1.12) suggest the following form for $I_{i,j}$

$$\frac{\mathrm{Tr}(\tilde{X} X_{i+1} \tilde{X}_i X_0 \tilde{X}_j X_{j+1})}{(X_0 \cdot X_i)(X_0 \cdot X_{i+1})(X_0 \cdot X_j)(X_0 \cdot X_{j+1})(X_0 \cdot X)} \tag{6.1.14}$$

Note that the Trace above (i.e. $\mathrm{Tr}(\cdot)$) simply means that we close the indices as follows $X_{\alpha\beta} \ldots X^{\cdot,\alpha}$, this corresponds to taking the trace if we were to write out our X's as matrices. Here in the integration over twistors A, B becomes integration over the X-space variable X_0. To see that this does indeed reproduce the terms we want we may simply use (6.1.12) to expand out the numerator

and see that after cancellations we are left only those terms we require. This is a particularly simple example and we find that it will no longer be so simple once we inspect higher loop-orders and more external points with lots of terms which may have cancellations between them. As an example of this we have no expression like (6.1) for MHV amplitudes with arbitrary numbers of external points at 3-loops. Were we to use dual-momenta coordinates we would find that (6.1) is equivalent to

$$\frac{\text{Tr}(\tilde{x}_{i,i+1}x_{i+1,0}\tilde{x}_{0,j}x_{j,j+1}\tilde{x}_{j+1,x}x_{x,i})}{x_{0,i}^2 x_{0,i+1}^2 x_{0,j}^2 x_{0,j+1}^2 x_{0x}^2} \tag{6.1.15}$$

The variable X is a reference twistor meaning it is an arbitrary value and should drop out of the sum which gives the one-loop amplitude, we shall see that the attempt to recover the full $\mathcal{O}(\varepsilon)$ answer will put restrictions on what choice we can make for this reference twistor.

As a last demonstration, we recast the one-loop hexagon

$$\int_{AB} \frac{\langle AB13\rangle\langle AB46\rangle\langle 5612\rangle\langle 2345\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB56\rangle\langle AB61\rangle} \tag{6.1.16}$$

which may be more familiar to some readers as:

$$\begin{aligned}\frac{\text{Tr}(\tilde{X}_3 X_0 \tilde{X}_1 X_2 \tilde{X}_6 X_0 \tilde{X}_4 X_5)}{(X_0{\cdot}X_1)(X_0{\cdot}X_2)(X_0{\cdot}X_3)(X_0{\cdot}X_4)(X_0{\cdot}X_5)(X_0{\cdot}X_6)}\\=\frac{\text{Tr}(\tilde{x}_{3,0}x_{0,1}\tilde{x}_{1,2}x_{2,6}\tilde{x}_{6,0}x_{0,4}\tilde{x}_{4,5}x_{5,3})}{x_{0,1}^2 x_{0,2}^2 x_{0,3}^2 x_{0,4}^2 x_{0,5}^2 x_{0,6}^2}\end{aligned} \tag{6.1.17}$$

6.2   THE MEANING OF μ^2

The first thing we must do is explain how our $([X_0]{\cdot}X_0)$ is to be proportional to μ^2 and in turn gives us our ε contribution? Using the particular method of embedding variables which we outlined (6.1.2), we dimensionally regulate the integration variable X_0 as follows:

$$X_0 = \begin{pmatrix} \frac{1-x_0^2}{2} \\ x_0^{\hat{\nu}} \\ \frac{1+x_0^2}{2} \end{pmatrix} \tag{6.2.1}$$

where $\hat{\nu}$ is a $4-2\epsilon$ dimensional index which takes part of the 4-dimensional piece in a Lorentz metric with the extra -2ε dimensional piece ν.

$$x_0^{\hat{\nu}} = \begin{pmatrix} x_0^i \\ \mu \end{pmatrix} \tag{6.2.2}$$

Note that this X_0 is zero when dotted with itself in a metric $(+,+,-,-,-,-)$ which is appropriately modified to $(4-2\varepsilon)$ dimensions, i.e. we find

$$X_0 \cdot X_0 = \left(\frac{1-x_0^2}{2}\right)^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2 - \mu^2 - \left(\frac{1-x_0^2}{2}\right)^2 \tag{6.2.3}$$

However, when its inside the trace or epsilon then it only makes sense to be in precisely 6-dimensions, as such we use $[X_0]$ to mean projected into 6-dimensions, i.e. drop the μ. Note $[X_0] \cdot X_0 \neq 0$ since we set the -2ε piece μ to zero in the vector but we still have μ inside the x_0^2 first and last entries. As such we obtain:

$$-x_0^{\hat{\nu}} x_{0,\hat{\nu}} + x_0^{\nu} x_{0,\nu} = \mu^2 \tag{6.2.4}$$

When we have the above integrals proportional to μ^2 we would like to simplify this by removing it from the numerator and aligning with previous literature such as [102] by moving from 4-dimensions into 6-dimensions, this section makes this transition explicit. Firstly we can break the integration over $4-2\epsilon$ dimensions as follows:

$$\int \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \rightarrow \int \frac{d^4p}{(2\pi)^4} \int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} \tag{6.2.5}$$

Any integral such as those above in which there is some power of μ^2 in the numerator is referred to as a μ-integral, which we now proceed to rewrite in a dimensionally shifted manner. To do this we manipulate the integration over μ as follows

$$\begin{aligned} \int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} f(\mu^2) \quad &= \int \frac{d\Omega_{-2\varepsilon}}{(2\pi)^{-2\varepsilon}} \int_0^\infty d\mu \mu^{-2\varepsilon-1} f(\mu^2) \\ &= \tfrac{1}{2} \int \frac{d\Omega_{-2\varepsilon}}{(2\pi)^{-2\varepsilon}} \int_0^\infty d\mu^2 (\mu^2)^{-\varepsilon-1} f(\mu^2) \end{aligned} \tag{6.2.6}$$

where the first step involves breaking the integration of the variable μ in -2ε-dimensions into a -2ε-dimensional sphere and the remaining integration over

the 'radius' μ. The area of a sphere in D=-2ε dimensions is given by:

$$\int d\Omega_{-2\varepsilon} = \frac{(2\pi)^{-2\varepsilon}}{\Gamma(-\varepsilon)} \tag{6.2.7}$$

Now if we moderate the argument of $f(\mu^2)$ by including μ^{2r} we can absorb the extra factors of μ^2 into the integration measure as follows:

$$\int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} \mu^{2r} f(\mu^2) \quad = \frac{(2\pi)^{2r} \int d\Omega_{-2\varepsilon}}{\int d\Omega_{2r-2\varepsilon}} \int \frac{d^{2r-2\varepsilon}\mu}{(2\pi)^{2r-2\varepsilon}} f(\mu^2)$$

$$= -\varepsilon(1-\varepsilon)...(r-1-\varepsilon)(4\pi)^r \int \frac{d^{2r-2\varepsilon}}{(2\pi)^{2r-2\varepsilon}} f(\mu^2) \tag{6.2.8}$$

So if r is a positive integer this analysis leads to the concise formula:

$$I_n^{D=4-2\varepsilon}[\mu^{2r}] = -\varepsilon(1-\varepsilon)(2-\varepsilon)...(r-1-\varepsilon)I_n^{D=2r+4-2\varepsilon} \tag{6.2.9}$$

To be precise what this means for our integrals with a factor of μ^2 in the numerator (which we will encounter later) is that these integrals can be thought of as being integrals in 6-dimensions but proportional to ε.

6.3   1 Loop MHV

We begin by considering the BCFW expression for a 1-loop MHV integrand expression with a single reference twistor as given above in (6.1) (equation (6.4) of [9]), which is given in the form of:

$$\sum_{i<j} \frac{\langle AB(i-1,i,i+1) \cap (j-1,j,j+1)\rangle\langle Xij\rangle}{\langle ABX\rangle\langle AB,i-1,i\rangle\langle AB,i,i+1\rangle\langle AB,j-1,j\rangle\langle AB,j,j+1\rangle} \tag{6.3.1}$$

We now recast this equation in 6d vectors $X_i = (X_{-1}^i, X_0^i, X_1^i, X_2^i, X_3^i, X_4^i)$, which are null in the Lorentz metric $(2,4)$. Relating $X_i^{\alpha\beta} = Z_i^\alpha Z_{i+1}^\beta - Z_i^\beta Z_{i+1}^\alpha$ and defining $\tilde{X}_{\alpha,\beta} = \epsilon_{\alpha\beta\gamma\delta}X^{\gamma\delta}$, we find that the above equation can be rewritten as:

$$\sum_{i<j} \frac{\text{Tr}(\tilde{X}_0 X_{i+1} \tilde{X}_i X \tilde{X}_j X_{j+1})}{(X_i \cdot X_0)(X_{i+1} \cdot X_0)(X \cdot X_0)(X_j \cdot X_0)(X_{j+1} \cdot X_0)} \tag{6.3.2}$$

However in the interest of trying to dimensionally regularize this equation we will consider the integration variable X_0 as having $6-2\varepsilon$ components, with all other X's still having the full 6-components (though naturally being restricted to a 4-dimensional space as can be seen in (6.1.2)).

Expanding out the parity-even part of the trace (leaving the parity-odd part for later):

$$\text{Tr}(\tilde{X}_0 X_{i+1} \tilde{X}_i X \tilde{X}_j X_{j+1})_{\text{even}} \tag{6.3.3}$$

$$= -(X_0 \cdot X) \left[(X_{i+1} \cdot X_j)(X_i \cdot X_{j+1}) - (X_i \cdot X_j)(X_{i+1} \cdot X_{j+1}) \right]$$

$$+ \left(\{ [(X_j \cdot X)(X_0 \cdot X_i)(X_{i+1} \cdot X_{j+1}) - (i \leftrightarrow i+1)] - (j \leftrightarrow j+1) \} + (i \leftrightarrow j) \right) \tag{6.3.4}$$

Then after cancellations we obtain only the sum over two-mass-easy boxes. The parity-odd component is more complicated

$$- \sum_{i<j} \frac{\epsilon(X_0, X_{i+1}, X_i, X, X_j, X_{j+1})}{(X_i \cdot X_0)(X_{i+1} \cdot X_0)(X \cdot X_0)(X_j \cdot X_0)(X_{j+1} \cdot X_0)} \tag{6.3.5}$$

From this point it becomes expedient to consider the above equation separately for a fixed number of external points. To this end we begin by limiting ourselves to 5-points only, before moving on to the 6-point case.

Parity-Odd at 5-points

Clearly if $j = i+1$ the epsilon will vanish, as such we get only 5 terms in the sum. We will be taking great care of our integration variables such as X_0, which lies in D-dimensions once we turn to dimensional regularization. Putting over a common denominator this equation becomes:

$$- \frac{\epsilon(X_0, X_1, X_2, X, X_3, X_4)(X_5 \cdot X_0) + \text{cyclic}\,\{1, 2, 3, 4, 5\}}{(X_1 \cdot X_0)(X_2 \cdot X_0)(X_3 \cdot X_0)(X_4 \cdot X_0)(X_5 \cdot X_0)(X \cdot X_0)} \tag{6.3.6}$$

Note that X_0 along with all other variables inside ϵ must lie in 6-dimensions for this tensor to be well-defined. From this we now use the identity for 7 arbitrary points Y_i, which are valid coordinates in 6-dimensions

$$\epsilon(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)Y_7 + \text{cyclic}\,\{1, 2, 3, 4, 5, 6, 7\} = 0 \tag{6.3.7}$$

and applying this to the above case yields the numerator:

$$\epsilon(X_1, X_2, X_3, X_4, X_5, X_0)(X \cdot X_0) + \epsilon(X, X_1, X_2, X_3, X_4, X_5)([X_0] \cdot X_0) \tag{6.3.8}$$

where we remind ourselves that $[X_0]$ means we project from $6 - 2\varepsilon$ components on to 6 components. Plugging this numerator into our overall equation and setting $([X_0] \cdot X_0) = \mu^2$, we find the parity-odd part of the integrand at 5-points

to be equal to:

$$\frac{\epsilon(X_0,X_1,X_2,X_3,X_4,X_5)}{(X_1 \cdot X_0)(X_2 \cdot X_0)(X_3 \cdot X_0)(X_4 \cdot X_0)(X_5 \cdot X_0)}$$
$$+ \frac{\mu^2 \epsilon(X_1,X_2,X_3,X_4,X_5,X)}{(X_1 \cdot X_0)(X_2 \cdot X_0)(X_3 \cdot X_0)(X_4 \cdot X_0)(X_5 \cdot X_0)(X \cdot X_0)} \tag{6.3.9}$$

Before we consider (6.3.5) at 6-points we should discuss how to interpret our above result. Notice firstly that by setting our reference twistor X to be any external twistor, we lose the second of the two terms above. This is our first example that once you dimensionally regularize the integrands the reference twistors may no longer be arbitrary. The first of the two terms is however X-independent and for generic external momenta will not vanish.

If we wish to match this term to known results then we have to solve the remaining term in the following way, we set $Y_a = \epsilon(X_1, X_2, X_3, X_4, X_5, \cdot) \equiv \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 + \beta I$ and solve for the coefficients by the series of simultaneous equations[11]:

$$Y_a \cdot X_i = 0 \qquad\qquad i = 1, ..., 5$$
$$Y_a \cdot I = \epsilon(X_1, X_2, X_3, X_4, X_5, I) = 4\alpha_1 + 4\alpha_2 + 4\alpha_3 + 4\alpha_4 + 4\alpha_5 := y \tag{6.3.10}$$

where we have used the fact that $X_i \cdot I = 4$ for any physical X_i including X_0. Then, once we have solved these equations we set the numerator to be $Y_a \cdot X_0$ which gives us

$$\alpha_1(X_1 \cdot X_0) + \alpha_2(X_2 \cdot X_0) + \alpha_3(X_3 \cdot X_0) + \alpha_4(X_4 \cdot X_0) + \alpha_5(X_5 \cdot X_0) + \beta y \tag{6.3.11}$$

The first five terms cancel with one of the five propagators giving five boxes and the last term leaves all six propagators and as such gives us a pentagon. We obtain solutions for the coefficients of the form $(s_i = X_i \cdot X_{i+2} \equiv \frac{1}{2}(p_i + p_{i+1})^2)$

$$\alpha_1 = \frac{-y s_2 s_3 (s_1 s_2 - s_2 s_3 + s_3 s_4 - s_4 s_5 + s_5 s_1)}{4 \sum_{i=1}^{5}(s_i^2 s_{i+1}^2 - 2 s_i s_{i+1}^2 s_{i+2} + 2 s_i s_{i+1} s_{i+2} s_{i+3})} \quad i \in \mod(5)$$
$$\beta = \frac{y s_1 s_2 s_3 s_4 s_5}{8 \sum_{i=1}^{5}(s_i^2 s_{i+1}^2 - 2 s_i s_{i+1}^2 s_{i+2} + 2 s_i s_{i+1} s_{i+2} s_{i+3})} \quad i \in \mod(5) \tag{6.3.12}$$

To compare these solutions to other results in the literature we need to recast these equations in terms of momentum variables.

We start by considering the denominator and make the ansatz $\mathrm{Det}(p_i \cdot p_j)$, where the s_i's are defined as above as being Mandelstam variables and i, j run

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[11] Note here that I is the so-called infinity-twistor and is as such not a physical point since it represents a point at infinity, this breaks dual conformal invariance.

over $1, ..., 4$.

$$\text{Det}(p_i \cdot p_j) = \begin{vmatrix} 0 & (p_1 \cdot p_2) & (p_1 \cdot p_3) & (p_1 \cdot p_4) \\ (p_1 \cdot p_2) & 0 & (p_2 \cdot p_3) & (p_2 \cdot p_4) \\ (p_1 \cdot p_3) & (p_2 \cdot p_3) & 0 & (p_3 \cdot p_4) \\ (p_1 \cdot p_4) & (p_2 \cdot p_4) & (p_3 \cdot p_4) & 0 \end{vmatrix}$$

$$= \frac{1}{2^4} \begin{vmatrix} 0 & (p_1+p_2)^2 & (p_1+p_3)^2 & (p_1+p_4)^2 \\ (p_1+p_2)^2 & 0 & (p_2+p_3)^2 & (p_2+p_4)^2 \\ (p_1+p_3)^2 & (p_2+p_3)^2 & 0 & (p_3+p_4)^2 \\ (p_1+p_4)^2 & (p_2+p_4)^2 & (p_3+p_4)^2 & 0 \end{vmatrix} \quad (6.3.13)$$

now we want to express all these terms in terms of s_i's alone. The key is in using momentum conservation $\sum_{i=1}^{5} p_i = 0$, note that as a result the following stage is special to 5-points. For example, recalling that all external momenta are null, we can express $p_1 \cdot p_3$ using the following series of equations:

$$\frac{1}{2}(p_1+p_2+p_3)^2 = \frac{1}{2}(p_4+p_5)^2$$
$$\frac{1}{2}(p_1+p_3)^2 + s_1 + s_2 = s_4 \quad (6.3.14)$$

we now rearrange this and insert it and equivalent equations to get:

$$\begin{vmatrix} 0 & s_1 & s_4-s_1-s_2 & s_2-s_4-s_5 \\ s_1 & 0 & s_2 & s_5-s_2-s_3 \\ s_4-s_1-s_2 & s_2 & 0 & s_3 \\ s_2-s_4-s_5 & s_5-s_2-s_3 & s_3 & 0 \end{vmatrix} \quad (6.3.15)$$

$$= \sum_{i=1}^{5} (s_i^2 s_{i+1}^2 - 2 s_i s_{i+1}^2 s_{i+2} + 2 s_i s_{i+1} s_{i+2} s_{i+3}) \quad (6.3.16)$$

so (up to a constant factor) this agrees with our denominator from all our α's and β. Precisely the same reasoning and mode of calculation can show:

$$y = \sqrt{\text{Det}\left[(I, X_1, X_2, X_3, X_4, X_5) \cdot (I, X_1, X_2, X_3, X_4, X_5)^T\right]}$$

$$= 4 \sqrt{\sum_{i=1}^{5} (s_i^2 s_{i+1}^2 - 2 s_i s_{i+1}^2 s_{i+2} + 2 s_i s_{i+1} s_{i+2} s_{i+3})} \quad (6.3.17)$$

Finally considering the numerator it is simple to find

$$\mathrm{Tr}(p_1 p_2 p_3 p_5) = [(p_1 \cdot p_2)(p_3 \cdot p_5) - (p_1 \cdot p_3)(p_2 \cdot p_5) + (p_1 \cdot p_5)(p_2 \cdot p_3)]$$

$$= - [s_1 s_2 - s_2 s_3 + s_3 s_4 - s_4 s_5 + s_5 s_1] \qquad (6.3.18)$$

Where this trace is as earlier taken over the matrix representation of this combination, but in practice simply means $p_{1,\alpha} \cdots p_5^{\cdot \alpha}$. As such, using the calculations shown above we can match coefficients and show that, for example, the box associated with α_1 can be written as:

$$\frac{-s_2 s_3 \mathrm{Tr}(p_1 p_2 p_3 p_5)}{\sqrt{-16 \mathrm{Det}(p_i \cdot p_j)}} \frac{1}{(p_0 + p_2)^2 (p_0 + p_3)^2 (p_0 + p_4)^2 (p_0 + p_5)^2} \qquad (6.3.19)$$

and our pentagon has the form,

$$\frac{-s_1 s_2 s_3 s_4 s_5}{\sqrt{-8 \mathrm{Det}(p_i \cdot p_j)}} \frac{1}{(p_0 + p_1)^2 (p_0 + p_2)^2 (p_0 + p_3)^2 (p_0 + p_4)^2 (p_0 + p_5)^2} \qquad (6.3.20)$$

There are several questions which immediately arise from these results. For example, what happens if rather than setting our reference twistor to be a physical external point we instead set it to be the infinity twistor I, or any other arbitrary point? Or could we have reached this point by simpler methods than this rather involved process, and how can we use these results to compare with expressions found elsewhere? It is to this last question we now turn.

Following the work done by Schabinger [103] we note that the above equations are all to be understood as being in 4-dimensions in what is referred to as the "geometric basis". If we now use the equation given below to move into the dual conformal basis of 4-dimensional boxes and a 6-dimensional pentagon, then a simpler comparison may be possible from our original formula. This utilises:

$$I_5^{D=4-2\varepsilon} = \frac{1}{2} \left[\sum_{j=1}^{5} C_j I_4^{(j),D=4-2\varepsilon} + 2\varepsilon C_0 I_5^{D=6-2\varepsilon} \right] \qquad (6.3.21)$$

Here $I_4^{(j),D=4-2\varepsilon}$ denotes a box missing the propagator $(X_j + X_0)^2$ but in $D=4-2\varepsilon$ dimensions, $I_5^{D=6-2\varepsilon}$ denotes a pentagon in $6-2\varepsilon$ dimensions and the coefficients C_0 and C_j are given as:

$$C_i = \sum_{j=1}^{n} S_{i,j}^{-1} \qquad C_0 = \sum_{i=1}^{n} C_i \qquad (6.3.22)$$

Which utilised the matrix $S_{i,j}$ defined as follows

$$S_{i,j} = \tfrac{1}{2}(p_i + ... + p_{j-1})^2 \quad i \neq j$$
$$S_{i,j} = 0 \qquad\qquad i = j \tag{6.3.23}$$

with i, j understood mod(n) and for more details on this work see Appendix A of [103]. With these equations we have a way of relating a certain combination of 4-dimensional boxes and pentagons (when we are in the so-called geometric basis) to boxes in 4-dimensions and pentagons in 6-dimensions (dual-conformal basis).

$$\varepsilon I_5^{D=6-2\varepsilon} = \frac{1}{C_0} \left(I_5^{D=4-2\varepsilon} - \frac{1}{2} \sum_{j=1}^n C_j I_4^{(j), D=4-2\varepsilon} \right) \tag{6.3.24}$$

Next we shall see that the parity-odd part of 5-points nicely maps into a single pentagon in 6-dimensions proportional to ε, those familiar with the full $\mathcal{O}(\varepsilon)$ 5-point MHV amplitude might expect this based on the fact that the parity-even part gives us all the 2-mass-boxes matching the known amplitude up to $\mathcal{O}(\varepsilon)$, see equation (48) of [103] for this matching.

Calculating the necessary coefficients in (6.3.24) we find

$$\frac{2s_1 s_2 s_3 s_4 s_5}{2\sqrt{\text{Det}(p_i \cdot p_j)}} I_5^{D=4-2\varepsilon} + \frac{s_2 s_3 \text{Tr}(p_1 p_2 p_3 p_5)}{\sqrt{\text{Det}(p_i \cdot p_j)}} I_4^{(1), D=4-2\varepsilon} - ... = \varepsilon\sqrt{\text{Det}(p_i \cdot p_j)} I_5^{D=6-2\varepsilon}$$
$$\tag{6.3.25}$$

By checking these coefficients against α_i with $i = 1, \ldots, 5$ and y those found in (6.3.12) we can see there is a map from the parity-odd part of our amplitude to a 6-dimensional pentagon with the pre-factor $\varepsilon\epsilon(p_1, p_2, p_3, p_4) \equiv \varepsilon\sqrt{\text{Det}(p_i \cdot p_j)}$ matching expectations from [102, 103] and elsewhere. The question now becomes could we have avoided the transitional calculations and gone straight from the parity-odd pentagon in 4-dimensional X-space to this 6-dimensional pentagon in momentum-space and obtained the correct pre-factor? We did not use $([X_0] \cdot X_0)$ above, since our reference twistor X we set to be an external twistor however from 6-points here onwards we cannot remove it in this way.

6.4 Parity-Odd at 6-Points

As we showed at the start of this chapter, the parity-even part of the n-point amplitude is obtained in a very simple way, however at 5-points we used many

constricting arguments in our investigation of the parity-odd part. So at 6-points:

$$\sum_{i<j} \frac{\mathrm{Tr}(X_0 X_{i+1} X_i X X_j X_{j+1})}{(X_i \cdot X_0)(X_{i+1}\cdot X_0)(X\cdot X_0)(X_j\cdot X_0)(X_{j+1}\cdot X_0)}$$

$$= \frac{\mathrm{Tr}(X_0 X_2 X_1 X X_4 X_5)(X_3\cdot X_0)(X_6\cdot X_0)}{(X_1\cdot X_0)(X_2\cdot X_0)(X_3\cdot X_0)(X_4\cdot X_0)(X_5\cdot X_0)(X_6\cdot X_0)(X\cdot X_0)} + 2\,\mathrm{others}$$

$$+ \frac{\mathrm{Tr}(X_0 X_2 X_1 X X_3 X_4)(X_5\cdot X_0)(X_6\cdot X_0)}{(X_1\cdot X_0)(X_2\cdot X_0)(X_3\cdot X_0)(X_4\cdot X_0)(X_5\cdot X_0)(X_6\cdot X_0)(X\cdot X_0)} + 5\,\mathrm{others} \qquad (6.4.1)$$

We restrict our attention to the parity-odd part where the $\mathrm{Tr} \to \epsilon$, and apply similar manipulations using 7-particle identities to express the above in terms only proportional to $([X_0]\cdot X_0)$ or independent of X. The results are very much analogous to those of the 5-point case and once we have worked through all the terms using necessary Schouten identities we end up with the following terms in the numerator:

$$-\frac{1}{2}([X_0]\cdot X_0)\left[(X_1\cdot X_0)\epsilon(X_2 X_3 X_4 X_5 X_6 X) + \mathrm{cycle}\,\{1,2,3,4,5,6\}\right]$$

$$+\frac{1}{2}\left[(X\cdot X_0)\epsilon(X_2 X_3 X_4 X_5 X_6 X_0) + \mathrm{cycle}\,\{1,2,3,4,5,6\}\right] \qquad (6.4.2)$$

Here the second term is analogous to the term we solved at 5-points, and is a function of only 5 of the 6 points in each term, thus we might expect solutions to arise similar to those at 5-points. The other set of terms will now NOT all disappear by setting X to be an external twistor. Indeed if we set X to be X_1 we will be left with an overall term $\frac{1}{2}([X_0]\cdot X_0)\epsilon(X_1 X_2 X_3 X_4 X_5 X_6)$, whereas if we set it to be X_6 we would have the same answer but with a minus sign at the front, this ambiguity has no obvious resolution. Moreover when we compare to the all ε expansion of the 6-point amplitude from [24] and work done subsequently, we find that the integral of a hexagon there has a parity-even prefactor $\mathrm{Tr}\,[p_1 p_2 p_3 p_4 p_5 p_6]$ where we currently find a parity-odd prefactor. This collection of inconsistencies seems to indicate that the original formula fails to capture the full ε behaviour of our amplitudes. As such we look elsewhere to a generalisation of this formula with two reference twistors also to be found in [9].

6.5   1 Loop MHV - 2 Reference Twistors

We take the form of the one-loop, planar, MHV amplitude with arbitrary number of external points from (equation 48) [8], which at least before dimensional regularisation is an equivalent function to that already considered, however this function has two reference twistors X and Y. The equation in question is:

$$M_{1-\text{loop,n}}^{\text{MHV}} = \sum_{i,j} \frac{\langle AB(Xi)\cap(Yj)\rangle\langle AB(i{-}1,i,i{+}1)\cap(j{-}1,j,j{+}1)\rangle}{\langle ABX\rangle\langle ABY\rangle\langle AB,i{-}1,i\rangle\langle AB,i,i{+}1\rangle\langle AB,j{-}1,j\rangle\langle AB,j,j{+}1\rangle}$$

(6.5.1)

where this numerator is understood to mean

$$(\langle AXi\rangle\langle BYj\rangle - (A \leftrightarrow B))(\langle A,i{-}1,i,i{+}1\rangle\langle B,j{-}1,j,j{+}1\rangle - (A \leftrightarrow B)) \quad (6.5.2)$$

In X's this numerator is simply $\text{Tr}(X_0\tilde{X}X_i\tilde{X}_{i+1}X_0\tilde{X}_{j+1}X_j\tilde{Y})$ which expanded back out gives

$$= X_{0,AB}\tilde{X}^{BC}X_{i,CD}\tilde{X}_{i+1}^{DE}X_{0,EF}\tilde{X}_{j+1}^{FG}X_{j,GH}\tilde{Y}^{HA}$$

(6.5.3)

with $\tilde{X}^{BC} = \frac{1}{2}\epsilon^{BC\alpha\beta}X_{\alpha\beta}$ etc. We decompose these bi-twistors into single twistor components, e.g. $X_{0,EF} = Z_{A,E}Z_{B,F} - Z_{A,F}Z_{B,E}$ and similarly for $X_{i,CD} = Z_{i,[C}Z_{i+1,D]}$ etc. Expanding this out we find that we are left with only four terms which match the original numerator as required.

So, we now proceed in a similar vein to the earlier case of a single reference twistor. This leaves us with only two terms, one independent of X and one dependent on both X and Y:

$$(X_0 \cdot X_{i+1})\text{Tr}(X\tilde{X}_i X_0 \tilde{X}_{j+1} X_j \tilde{Y}) - \frac{1}{2}([X_0]\,X_0)\text{Tr}(X\tilde{X}_i X_{i+1}\tilde{X}_{j+1}X_j\tilde{Y})$$

(6.5.4)

In particular, the first term above is identical to the formula we were analysing in the previous section (though with a minus sign) and we know how that acts at 5- and 6-points. As such let's limit ourselves to the last term above. If we again impose the restriction that both X and Y are external twistors then we ensure there is no parity-odd part until 6-points.

The first thing we will do is consider these twistors to be $X \to X_{k+1}$ and $Y \to X_k$. Looking at the parity-even part at 5-points we can easily see there

are no contributions until at least 6-points. Let us suppose we had not fixed X and Y in such a nice way then we may have terms at six-points and beyond such as $\mathrm{Tr}(X_5, X_3, X_2, X_5, X_4, X_1)$ this will contribute zero at 5-points, however will give a non-zero result at 6-points once $(X_1{\cdot}X_5) \neq 0$ and as such we see that, setting X and Y to be external twistors alone is not sufficient to generate the correct solution as we now show.

Consider 6-points where we will have two terms:

$$-([X_0]{\cdot}X_0)\mathrm{Tr}_{\text{parity}-\text{even}}(X_1 X_2 X_3 X_4 X_5 X_6)-([X_0]{\cdot}X_0)\epsilon(X_1 X_2 X_3 X_4 X_5 X_6)$$
$$(6.5.5)$$

this parity-odd term then cancels with the parity-odd part from before $(Y \to X_k :=X_6)$ which we recall has a minus sign in the above expression and a factor of 2 since in this new expression we have no restriction that $i < j$. As such, it looks very much like we have achieved the correct expressions at 5-points and 6-points, we need only to check the coefficients for the parts proportional to ε and show

$$\mathrm{Tr}(X_1, X_2, X_3, X_4, X_5, X_6)=\mathrm{Tr}(p_1, p_2, p_3, p_4, p_5, p_6) \qquad (6.5.6)$$

which can be found very easily to be the case and to be equivalent to

$$\mathrm{Tr}(p_1, p_2, p_3, p_4, p_5, p_6)=(s_1 t_2 s_4+t_1 s_6 s_3+s_5 s_2 t_3-t_1 t_2 t_3) \qquad (6.5.7)$$

the μ^2 coefficient gives us the $-\varepsilon$, which results in a perfect matching.

So to be explicit, the key result we have shown in this subsection is that the part of the expression proportional to ε precisely matches the expected part from [102, 103] at 5- and 6-points. As such, assuming the rest matches as postulated by [9] we have the result

$$A_{\mathrm{MHV}}^{1-\mathrm{loop}}=\sum_{i,j} \frac{\mathrm{Tr}(X_0 \tilde{X}_{k+1} X_i \tilde{X}_{i+1} X_0 \tilde{X}_{j+1} X_j \tilde{X}_k)}{(X_0{\cdot}X_k)(X_0{\cdot}X_{k+1})(X_0{\cdot}X_i)(X_0{\cdot}X_{i+1})(X_0{\cdot}X_j)(X_0{\cdot}X_{j+1})} \quad (6.5.8)$$

where k is fixed. Note we could easily sum over k also and divide by n, which yields a slightly more symmetrical answer, and possibly this could be necessary for 7-points or above. The above result is only shown explicitly for 6-points, and modifications to the reference-twistor prescription cannot be forbidden without further research in the area. Let us also make explicit that thus far we have never been given the $\mathcal{O}(\varepsilon)$ terms by the expressions involving reference twistors without restrictions. That is we have always been required to make the "correct" choice for the reference twistors and as such we now attempt the

same processes as above but for the two-loop equation which has no reference twistors.

6.6   2-loop MHV

We wish to repeat this process again but for the two-loop amplitude, as such we again start with an amplitude in momentum twistors as given in [9]:

$$
\frac{1}{2} \sum_{i<j<k<l<i} \frac{\langle AB(i{-}1,i,i{+}1) \cap (j{-}1,j,j{+}1)\rangle}{\langle AB,i{-}1,i\rangle \langle ABi,i{+}1\rangle \langle AB,j{-}1,j\rangle \langle AB,j,j{+}1\rangle}
$$
$$
\times \frac{\langle i,j,k,l\rangle \langle CD(k{-}1,k,k{+}1) \cap (l{-}1,l,l{+}1)\rangle}{\langle ABCD\rangle \langle CD,k{-}1,k\rangle \langle CD,k,k{+}1\rangle \langle CD,l{-}1,l\rangle \langle CD,l,l{+}1\rangle}
$$
$$(6.6.1)$$

At 5-points we have the advantage that we are continually considering 'boundary cases' [12] where either $j{=}i{+}1$ and $k{=}j{+}1$ (or indeed any other pair of 'boundaries'). Indeed, at 6-points we have a similar result, which is that we always have one of three boundaries where the three we choose is irrelevant. Let us begin then by considering how to express in a single trace or a double-trace the boundary-term $j{=}i{+}1$, we do this to find:

$$
\frac{1}{2} \frac{\mathrm{Tr}(X_{AB}\tilde{X}_{i+1}X_i\tilde{X}_{i+2})}{(X_{AB}{\cdot}X_i)(X_{AB}{\cdot}X_{i+1})^2(X_{AB}{\cdot}X_{i+1})(X_{AB}{\cdot}X_{CD})}
$$
$$
\times \frac{\mathrm{Tr}(X_{CD}\tilde{X}_k X_{k+1}\tilde{X}_{i+1}X_{l+1}\tilde{X}_l)}{(X_{CD}{\cdot}X_k)(X_{CD}{\cdot}X_{k+1})(X_{CD}{\cdot}X_l)(X_{CD}{\cdot}X_{l+1})}
$$
$$(6.6.2)$$

Here we use indices in latin letters to indicate integration variables i.e. $X_{AB} = Z_{[A}Z_{B]}$ and we will integrate over the twistors Z_A, Z_B. We obtain an analogous double-trace structure for the boundary $l{=}k{+}1$. Whereas for the boundary where $k{=}j{+}1$ we find a term with numerator:

$$
\frac{1}{2}\mathrm{Tr}(X_{AB}\tilde{X}_i X_{i+1}\tilde{X}_{j+1}X_{l+1}\tilde{X}_l X_{CD}\tilde{X}_{j+2}X_{j+1}\tilde{X}_{j+1})
$$
$$(6.6.3)$$

---

[12] For arbitrary numbers of external particles we have no method for writing out the numerator in traces, we require one pair of i, j, k, l to differ by one to allow us to write a term in trace-structure. As such our first problem term emerges at 8-points when we have external-particles and $i{=}1$, $j{=}3$, $k{=}5$ and $l{=}7$ or any cyclically related term.

and denominator

$$(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})(X_{AB}\cdot X_j)(X_{AB}\cdot X_{j+1})(X_{AB}\cdot X_{CD})$$
$$(X_{CD}\cdot X_{j+1})(X_{CD}\cdot X_{j+2})(X_{CD}\cdot X_l)(X_{CD}\cdot X_{l+1}) \tag{6.6.4}$$

We now express the 5-point amplitude in terms of the two boundary terms $i+1{=}j$ and $k+1{=}l$ but with the use of the notation $i \lhd j$ to mean $i+1 < j$ to prevent double-counting of the case where both boundaries are in operation. As such we have the claim:

$$A_{\text{MHV}}^{2-\text{loops}}=$$

$$\sum_{i<i+1<k\lhd l<i} \frac{\text{Tr}(X_{AB}\tilde{X}_{i+1}X_i\tilde{X}_{i+2})}{(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})^2(X_{AB}\cdot X_{i+2})(X_{AB}\cdot X_{CD})}$$

$$\times \frac{\text{Tr}(X_{CD}\tilde{X}_kX_{k+1}\tilde{X}_{i+1}X_{l+1}\tilde{X}_l)}{(X_{CD}\cdot X_k)(X_{CD}\cdot X_{k+1})(X_{CD}\cdot X_l)(X_{CD}\cdot X_{l+1})}$$

$$+ \sum_{i<j<k<k+1<i} \frac{\text{Tr}(X_{CD}\tilde{X}_{k+1}X_k\tilde{X}_{k+2})}{(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})(X_{AB}\cdot X_j)(X_{AB}\cdot X_{j+1})(X_{AB}\cdot X_{CD})}$$

$$\times \frac{\text{Tr}(X_{AB}\tilde{X}_iX_{i+1}\tilde{X}_{k+1}X_{j+1}\tilde{X}_j)}{(X_{CD}\cdot X_k)(X_{CD}\cdot X_{k+1})^2(X_{CD}\cdot X_{k+2})} \tag{6.6.5}$$

We now look to expand this equation at 5-points into parity-even and parity-odd parts, note that the parity-odd part bears a remarkable resemblance to the 1-loop MHV solution given earlier.

$$A_{\text{MHV,5}}^{2-\text{loops}}=$$

$$\frac{(X_1\cdot X_3)\text{Tr}_{\text{P.E.}}(X_{CD},X_1,X_2,X_3,X_4,X_5)}{(X_{AB}\cdot X_1)(X_{AB}\cdot X_2)(X_{AB}\cdot X_3)(X_{AB}\cdot X_{CD})(X_{CD}\cdot X_3)(X_{CD}\cdot X_4)(X_{CD}\cdot X_5)}$$

$$- \frac{\epsilon(X_{CD}X_1X_2X_3X_4X_5)}{(X_{AB}\cdot X_1)(X_{AB}\cdot X_2)(X_{AB}\cdot X_3)(X_{AB}\cdot X_{CD})}$$

$$\times \frac{(X_1\cdot X_3)}{(X_{CD}\cdot X_3)(X_{CD}\cdot X_4)(X_{CD}\cdot X_5)(X_{CD}\cdot X_1)}+\text{cyclic} \tag{6.6.6}$$

If we expand out the parity-even part of this solution, then alongside the parity-odd part we have 5 distinct terms which appear before cyclicity. We show these terms in Fig.6.1, note the dotted lines indicate a contraction to be put in the numerator and a crossed circle centred on a integration variable denotes an epsilon which includes that variable and the other external variables. We can very easily now see that these match the diagrams drawn in Fig.2 of [22] and as such matches the 5-point two-loop amplitude found there up to coefficients
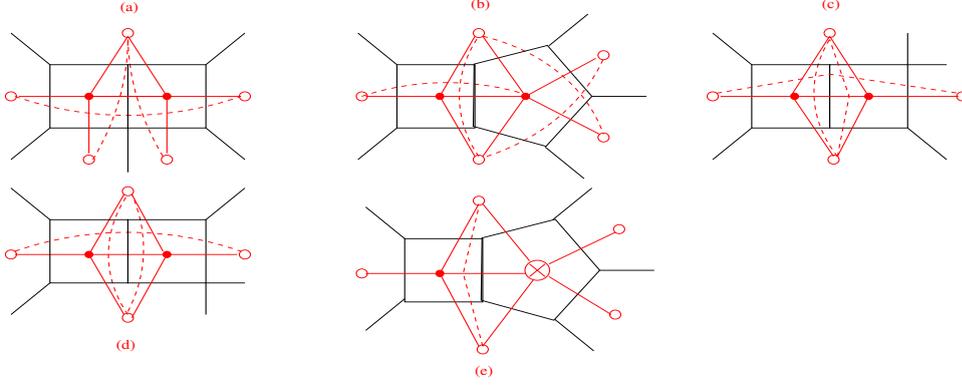
Figure 6.1: All the diagrams which contribute to the five-point 2-loop MHV amplitude, drawn both as standard (black) and dual-graphs (red). Note that (c) and (d) have equivalent dual-graphs due to the fact they are parity-conjugate to one another.

which we now turn our attention to. To check this, take the terms of (6.6.6) and expand them out in all possible manners and this will match the dual-graphs (red) in Fig.6.1 up to cyclic ordering of the indices. The standard (black) graphs are included throughout as they may be more familiar to most readers.

We use methods analogous with those we used to expand the $\epsilon(X_1, X_2, X_3, X_4, X_5, \cdot)$ from the integral $I^{(2)}_{(e)}(\varepsilon)$ into a basis of X_i's and the infinity twistor I (as we did in Sec.6.3). This removes this integral from our expansion and leaves us with only the other four. With this approach we match exactly the result from [22]:

$$M_5^{(2)}(\epsilon) = \frac{1}{8} \sum_{\text{cyclic}} \left\{ s_{12}^2 s_{23} I^{(2)}_{(d)}(\varepsilon) + s_{12}^{(2)} s_{15} I^{(2)}_{(c)}(\varepsilon) + s_{12} s_{34} s_{45} I^{(2)}_{(b)}(\varepsilon) \right. \tag{6.6.7}$$

$$\frac{s_{12} s_{23} s_{34} s_{45} s_{51}}{\sqrt{\text{Det}(p_i \cdot p_j)}} \left[2 I^2_{(a)}(\varepsilon) - 2 s_{12} I^{(2)}_{(b)}(\varepsilon) \right.$$

$$\left. + \frac{s_{12}}{s_{34} s_{45}} \left(\frac{\delta_{-++}}{s_{23}} I^{(2)}_{(c)}(\varepsilon) - \frac{\delta_{-+-}}{s_{51}} I^{(2)}_{(d)}(\varepsilon) \right) + \frac{\delta_{+-+}}{s_{23} s_{51}} I^{(2)}_{(b)}(\varepsilon) \right] \right\}$$

where $\delta_{abc} = s_{12} s_{51} + a s_{12} s_{23} + b s_{23} s_{34} - s_{51} s_{45} + c s_{34} s_{45}$. Once again at 5-points there are no explicit dual-conformally invariant terms which are proportional to μ^2. As such to find evidence that we can obtain such terms without further work through this method we will need to progress to the case of the 6-point, 2-loop MHV amplitude case where we expect that we will have some terms in the geometric basis which will be proportional to μ^2.

6.7   MHV 2-loops 6-point Amplitude

Now looking at the 6-point, two-loop MHV amplitude we may take the terms but with a suitably modified summation from the previous calculation and additionally add the extra term where $i+1 \neq j$ and $k+1 \neq l$. This leaves us with only one option at six points $j=i+2$, $k=i+3$ and $l=i+5$ and putting this into the 4-bracket archetype and transforming to X-notation we find the extra term with numerator:

$$\frac{1}{2}\text{Tr}(X_{AB}\tilde{X}_{i+2}X_{i+3}\tilde{X}_iX_{i+3}\tilde{X}_{i+4}X_{CD}\tilde{X}_{i+5}X_i\tilde{X}_{i+1}) \tag{6.7.1}$$

and denominator

$$(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})(X_{AB}\cdot X_j)(X_{AB}\cdot X_{j+1})(X_{AB}\cdot X_{CD})$$
$$\times (X_{CD}\cdot X_k)(X_{CD}\cdot X_{k+1})(X_{CD}\cdot X_l)(X_{CD}\cdot X_{l+1}) \tag{6.7.2}$$

Where we sum these terms over i and this allows us to write the following

$$A_{\text{MHV},6}^{2-\text{loop}}$$
$$=\frac{1}{2}\left(\sum_{i<i+1<k\triangleleft l<i \mod(6)} \frac{(X_i\cdot X_{i+2})}{(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})^2(X_{AB}\cdot X_{i+2})(X_{AB}\cdot X_{CD})} \right.$$
$$\times \frac{\text{Tr}(X_{CD}\tilde{X}_kX_{k+1}\tilde{X}_{i+1}X_{l+1}\tilde{X}_l)}{(X_{CD}\cdot X_k)(X_{CD}\cdot X_{k+1})(X_{CD}\cdot X_l)(X_{CD}\cdot X_{l+1})}$$
$$+ \sum_{i<j<k<k+1<i \mod(6)} \frac{(X_k\cdot X_{k+2})}{(X_{AB}\cdot X_i)(X_{AB}\cdot X_{i+1})(X_{AB}\cdot X_j)(X_{AB}\cdot X_{j+1})}$$
$$\left. \times \frac{\text{Tr}(X_{AB}\tilde{X}_iX_{i+1}\tilde{X}_{k+1}X_{j+1}\tilde{X}_j)}{(X_{AB}\cdot X_{CD})(X_{CD}\cdot X_k)(X_{CD}\cdot X_{k+1})^2(X_{CD}\cdot X_{k+2})} + \sum_{i \mod(6)} \frac{(6.7.1)}{(6.7.2)} \right) \tag{6.7.3}$$

If we expand out as done in earlier sections and use Schouten identities and dimensionally regularise X_{CD} into $(6-2\varepsilon)$-dimensions then we have 6 terms all related by cyclicity at $\mathcal{O}(\varepsilon)$:

$$\frac{\mu^2(X_1\cdot X_3)(X_{CD}\cdot X_2)\epsilon(X_1,...,X_6)}{(X_{AB}\cdot X_1)...(X_{AB}\cdot X_3)(X_{AB}\cdot X_{CD})(X_{CD}\cdot X_1)...(X_{CD}\cdot X_6)}+\text{cyclic} \tag{6.7.4}$$

these are contributions from a hexagon-box integral as in Fig.6.2. Having
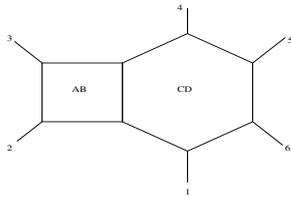
Figure 6.2: A hexagon-box integral of the type which gives us our terms proportional to ε

established the form of one of the possible $\mathcal{O}(\varepsilon)$ terms at 6-points we note that we have a huge variety of ways of expressing these terms due to relations such as Schouten identities.

If we write the terms directly out from (6.3) without using any relations used at all we obtain the graphs shown in Fig.6.3. We only have three coefficients α, β and γ which are integers relating to the symmetry in each of the three terms in (6.3). Note also that every graph in the figure which has a parity-conjugate not cyclically related to itself comes out with an identical pre-factor for both parity contributions, not necessarily non-trivial.

Note that we find $\alpha=\beta$ which comes naturally since the only difference between the two original terms is a relabelling and a difference in the restrictions in the sum which could be exchanged. As such we are reduced to two integer coefficients α an γ and the integral I_d now comes with a zero coefficient as predicted in [22], this again is a non-trivial check and we now match the form in momentum-space with that given in the same reference.

Let us now consider the coefficients and in particular start with the 1-loop^2 terms I_u, I_v and I_w along with their parity conjugates. At first glance these terms seem to have a simple cancellation where $(X_{AB} \cdot X_{CD})$ in the numerator cancels the same term in the denominator. However once we dimensionally regularise then this cancellation is no longer so simple. The denominator has the complete D-dimensional variables whereas the numerator requires them to be in 6-dimensions due to them being inside the trace. So our variables are schematically of the following form:

$$\begin{pmatrix} [X_{AB}] \\ \mu_{AB} \end{pmatrix} \tag{6.7.5}$$
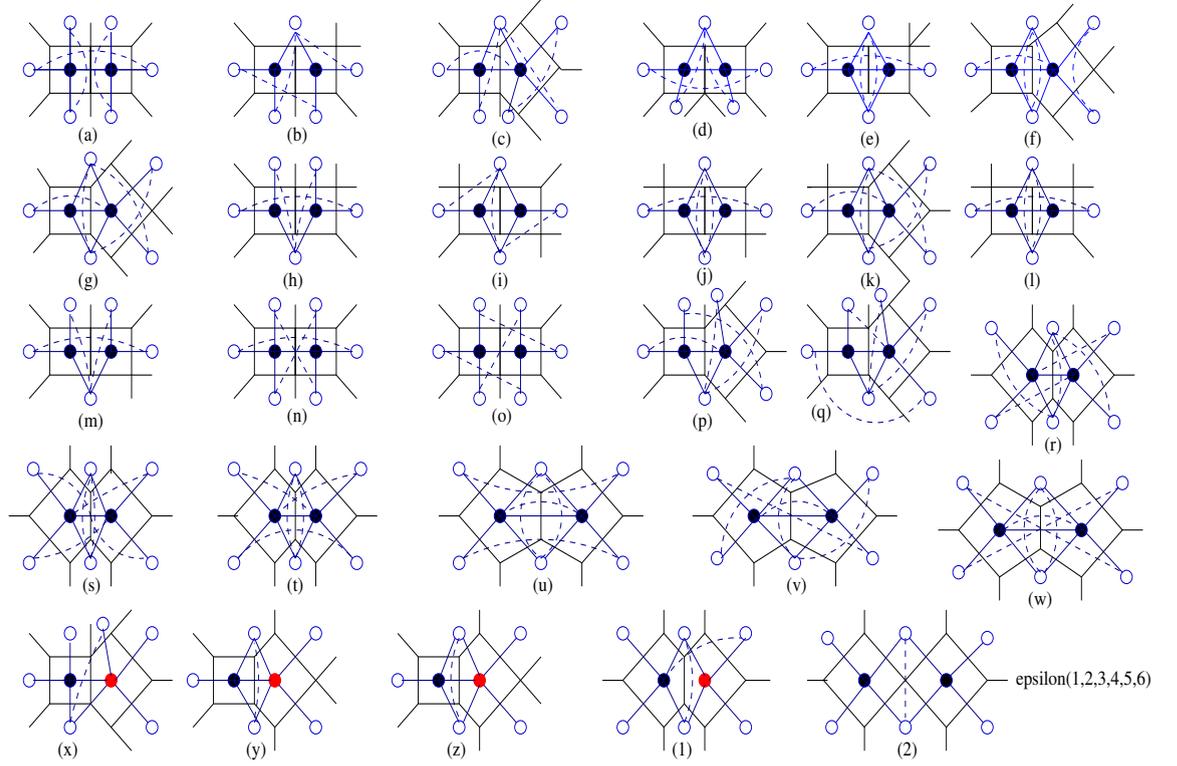
Figure 6.3: The set of all graphs given by expanding out our expression for the 6-point 2-loop MHV amplitude. Blue graphs are those we obtain by expanding out and the black graphs are the dual-graphs which will be more familiar to most readers.

So the ratio $\frac{[X_{AB}] \cdot [X_{CD}]}{X_{AB} \cdot X_{CD}}$ can be realised as $1 + \frac{\mu_{AB} \cdot \mu_{CD}}{X_{AB} \cdot X_{CD}}$. These terms give us the following contribution proportional to $1 + \mu_{AB} \cdot \mu_{CD}$:

$$\frac{(X_1 \cdot X_4)^2 (X_2 \cdot X_6)(X_3 \cdot X_5) + (X_1 \cdot X_4)(X_2 \cdot X_4)(X_3 \cdot X_6)(X_5 \cdot X_1)}{(X_{AB} \cdot X_1)(X_{AB} \cdot X_2)(X_{AB} \cdot X_3)(X_{AB} \cdot X_4)(X_{CD} \cdot X_4)(X_{CD} \cdot X_5)(X_{CD} \cdot X_6)(X_{CD} \cdot X_1)}$$
$$+ \frac{(X_1 \cdot X_4)(X_1 \cdot X_3)(X_2 \cdot X_4)(X_4 \cdot X_6) - (X_1 \cdot X_4)^2 (X_2 \cdot X_5)(X_3 \cdot X_6)}{(X_{AB} \cdot X_1)(X_{AB} \cdot X_2)(X_{AB} \cdot X_3)(X_{AB} \cdot X_4)(X_{CD} \cdot X_4)(X_{CD} \cdot X_5)(X_{CD} \cdot X_6)(X_{CD} \cdot X_1)} + \text{cyclic}$$

$$(6.7.6)$$

which it is then trivial to match to the coefficients c_1 and c_{14} in [28]. All these coefficients can be seen to match those in Equation (3.13) using Fig.3 in [28] except from their term with coefficient c_{15}, which is a hexa-box proportional to μ^2. However we also obtain the parity-conjugates to certain terms in $c_9 - c_{11}$ not explicitly mentioned in this reference. Nonetheless this is sufficient to demonstrate that our parity-even result matches all parts except the hexagon-box term proportional to μ^2.

As such, although we can draw comparisons between standard terms and $\mathcal{O}(\varepsilon)$-terms they do not come naturally and completely from the standard momentum-twistor expressions we have started from. In particular if we take diagrams $I_a^{(2)}$, $I_n^{(2)}$ and $I_o^{(2)}$ we can add propagators and numerators to cancel these to these diagrams to obtain a hexa-box. Having added for example $\frac{(X_{AB}\cdot X_1)(X_{AB}\cdot X_3)}{(X_{AB}\cdot X_1)(X_{AB}\cdot X_3)}$ using a trace we rearrange the numerator to give $\frac{([X_{AB}]\cdot X_{AB})(X_1\cdot X_3)}{(X_{AB}\cdot X_1)(X_{AB}\cdot X_3)}$ as an extra term and this gives us our missing μ^2-term. However this is merely fixing the result by hand to match known results, we shall now try and expand on the similarities between known terms and missing μ^2-terms.

6.8   Bridge between finite objects and $\mathcal{O}(\epsilon)$ terms

Let us now consider the form of the dual-conformally invariant μ^2-terms we have thus far obtained and attempt to form restrictions on their form. At one-loop with five external points, we found there was no dual-conformally invariant μ^2 contribution. This could have very easily been guessed since the full denominator of all physical poles would be $(X_0\cdot X_1)(X_0\cdot X_2)\dots(X_0\cdot X_5)$ and as such the numerator only has a single X_0 to ensure the correct conformal weight, as such it is obvious no $(X_0\cdot X_0)$-style term can appear in a planar dual-conformally invariant fashion. At two-loops with five external points we can have a 'complete' denominator

$$(X_0\cdot X_1)\cdots(X_0\cdot X_5)(X_0\cdot X_{0'})(X_{0'}\cdot X_1)\cdots(X_{0'}\cdot X_5) \tag{6.8.1}$$

note however that this is naively non-planar and as such some denominator lines must be cancelled in the numerator for planarity to be ensured. The numerator will have two X_0's and two $X_{0'}$'s which could conceivably contract together to give a μ^2 or $\mu_{AB}\cdot\mu_{CD}$ term. However we will now see that no such planar graph can have only 5 external legs.

Let us then check that neither a pentagon-pentagon or hexabox diagram can be drawn with only five external legs. Note that every vertex must naturally be of valency ≥ 3. As can be seen from Fig.6.4 we see that we require at least 6 external legs for either of these options to be viable and as such, in similarity to the earlier cases there will not be enough integration variables in

the numerator to allow μ^2 or $\mu_{AB}{\cdot}\mu_{CD}$ terms to be present while maintaining dual-conformal invariance.
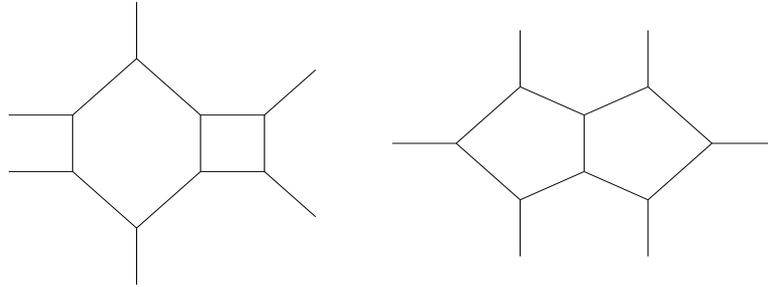
Figure 6.4: With all vertices of valency 3 these are the first graphs at two-loops which can give a μ^2 contribution and both require 6 or more external particles.

What about at 3-loops? We draw graphs of only 3-valent vertices since these will offer the largest number of free variables in the numerator and we find there is only two diagrams with five external particles which can contribute either a μ^2 or $\mu_{AB}{\cdot}\mu_{CD}$ term for 5-point, 3-loops as shown in Fig.6.5
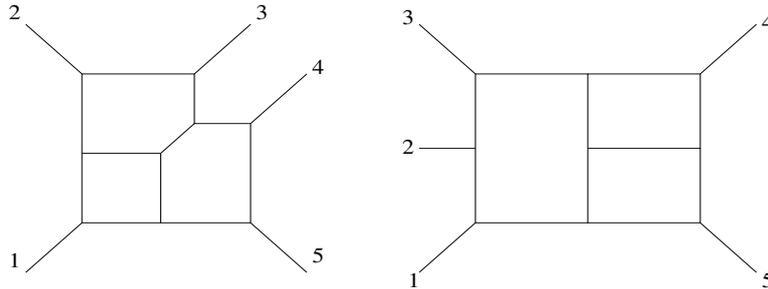
Figure 6.5: The first diagram could potentially contribute a $\mu_{AB}{\cdot}\mu_{CD}$-term and the second could contribute a dual-conformally invariant μ^2-term

The first term of Fig.6.5 (penta-penta-box) could only come with a numerator term $(X_0{\cdot}X_{0'})(X_1{\cdot}X_3)(X_1{\cdot}X_4)(X_2{\cdot}X_4)(X_2{\cdot}X_5)$ and the second diagram (hexa-box-box) with $(X_0{\cdot}X_0)(X_1{\cdot}X_4)^2(X_2{\cdot}X_5)(X_3{\cdot}X_5)$. For these to give μ^2 contributions we must assume everything in the numerator is in fact projected into four-dimensions, that is "they come from within a trace". Note in particular that the first diagram comes with both a standard and a μ^2 part. That is, there is a transformation shown in Fig.6.6 whereby the contraction $(X_0{\cdot}X_{0'})$ in the numerator gives a contribution which cancels the same bracket in the denominator and an extra part $\mu_{AB}{\cdot}\mu_{CD}$ where the propagator line remains. As such, these can be seen as the only potential terms for $\mathcal{O}(\epsilon)$ terms at 3-loops
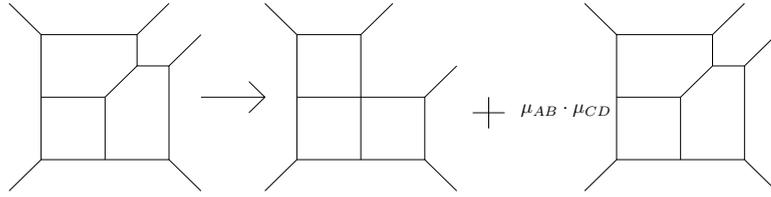
Figure 6.6: Note that the last term has no X_0 or $X_{0'}$ in the numerator but otherwise the numerator should correspond precisely with that of the middle term.

with 5 external particles and we have made a prediction for the coefficient of both of these diagrams

If we recall our results up until this point for six external particles we may see things which are analogous to our above analysis. The most general denominator at one-loop describes a hexagon $(X_0 \cdot X_1) \cdots (X_0 \cdot X_6)$ which could then potentially come with a numerator term $(X_0 \cdot X_0) \text{Tr}(X_1 \ldots X_6)$ where for reasons of cyclic symmetry the parity-odd part of this numerator vanishes and we only are allowed the parity-even part which is precisely what we did find. At six-points two-loops we found the penta-penta diagram and hexabox diagram already drawn in Fig.6.4 precisely as we could have anticipated and nothing additional could have been obtained as can easily be seen from trying to draw more elaborate diagrams.

However, in the examples considered the coefficients so far found are far from arbitrary, let us first consider the one-loop example. The process is indi-
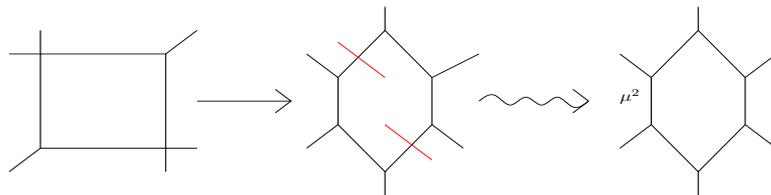
Figure 6.7: Here we demonstrate a way to see the relation between coefficients of finite-terms and terms giving a μ^2 contribution. The first step is to pull 4-valent vertices into 3-valent vertices by inserting propagators and corresponding numerator terms. Then place these into a trace such that before dimensional regularization you obtain only the necessary terms. Then after dimensional regularization you obtain in addition to all previously found terms a μ^2 term.

cated in Fig.6.7 namely we add in denominator lines and cancelling numerator terms to turn every 4-valent vertex into a pair of 3-valent vertices and then put

these numerator terms into the correct sum of traces. At this point and before
dimensional regularisation only the finite terms we want to get will come out
from the expansion of terms from within the trace. However after dimensional
regularisation we get a new and additional term which is proportional to μ^2
and shown on the right of Fig.6.7, however we have had to add this term by
hand according to the prescription shown in Fig.6.7.

At two-loops and six external particles an analogous thing occurs once
again, firstly the penta-penta diagram contributes a finite part where $(X_{AB}\cdot X_{CD})$
cancels from the denominator but also a term proportional to $\mu_{AB}\cdot\mu_{CD}$ as pic-
tured in Fig.6.8 and once again where the last term does not contain any
additional X_{AB} or X_{CD} in its numerator. The hexabox contribution with six
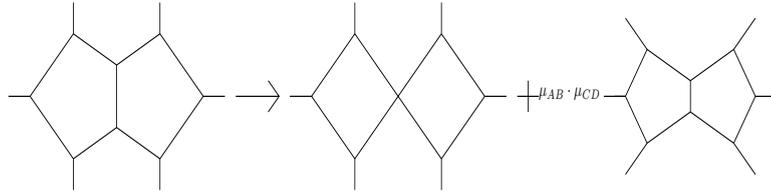
Figure 6.8: Here we again see a contraction of X_{AB} and X_{CD} gives both a
standard contribution and a dual-conformally invariant μ^2-term.

external particles can be seen as a contribution coming from the double-box
diagram but with similar manipulations to the two-mass-easy box shown in
Fig.6.7. That is we again insert cancelling numerator/denominator terms to
make all 4-valent vertices into pairs of 3-valent vertices, put the numerator
into traces whereby prior to dimensional regularisation we obtain only the cor-
rect terms and the dimensionally regularize to obtain the term proportional
to μ^2. Such manipulations are shown in Fig.6.9, but note yet again we have
not derived this term without intervention. We propose that such a technique,
though not prescriptively defined, may provide a way to attempt to generate
missing terms.
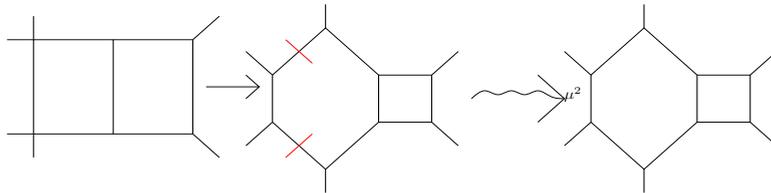
Figure 6.9: Manipulations to obtain a term proportional to μ^2 in precise anal-
ogy to the work shown in Fig.6.7

Up until this point these methods can correctly predict the dual-conformally invariant μ^2-terms with the correct coefficients [28, 102, 103]. We now therefore use this method which has an easy geometrical implication which we have demonstrated in the previous few Figures and we will use it to make predictions for potential $\mathcal{O}(\epsilon)$ terms for n-point ℓ-loop amplitudes. We start by seeing when the first possible μ^2-terms for 4-point amplitudes may occur.

The first possible μ^2 contributions to four-point amplitudes occur at four-loops as shown in Fig.6.10 where a contraction across the pentagons shown there gives a finite contribution and a $\mu_{AB} \cdot \mu_{CD}$ contribution both with $(X_1 \cdot X_3)^2$ $(X_2 \cdot X_4)^2$ in the numerator to complete the correct conformal weights. In
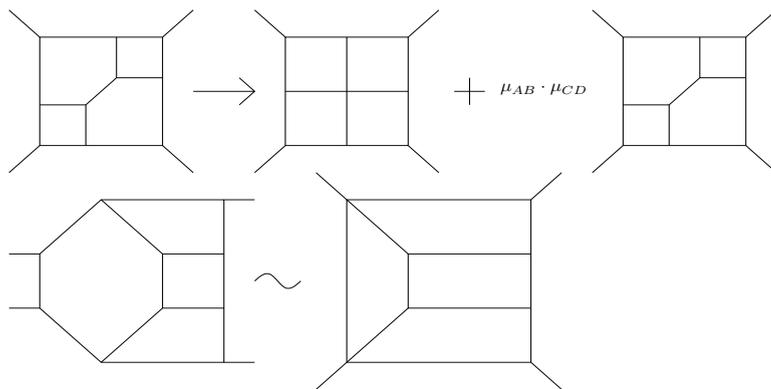
Figure 6.10: Possible terms at four-loops with four external particles which could contribute at $\mathcal{O}(\epsilon)$.

Fig.6.10 there is also a hexagon attached to three boxes which could conceivably give a μ^2 contribution, however it is simple to convince oneself that the numerator terms could not be contracted in such a way as to give a non-zero result. As such only the first diagram would be a potential μ^2-term which contributes. However in [16] the authors claim that $\mathcal{N}=4$ SYM amplitudes of four-particle amplitudes have no dual-conformally invariant μ^2-terms through four-loops. As such this method may not provide only terms which do in fact contribute and as such only constructs possible μ^2-terms which would then make a highly restricted ansatz. Ideally we could now discover some process whereby we could fix which terms from this ansatz do in fact contribute.

We finish this section by outlining once again the most general method for obtaining all graphs which could potentially give a dual-conformally invariant μ^2-term contribution. The key is to draw all planar graphs which are constructed from only 3-valent vertices. Any of these graphs which have a hexagon or more than one pentagon can potentially contribute either a μ^2 or

a $\mu_{AB}\cdot\mu_{CD}$ term. There may be certain diagrams such as the second example in Fig.6.10 where there is no non-zero way to contract the remainder of the numerator terms but this would become apparent quickly. This exceptionally simple method gives us for example 3-possible "master graphs" for five-points at four-loops as shown in Fig.6.11 where the contractions to get μ^2-term contributions can be done in several ways from the same graph.
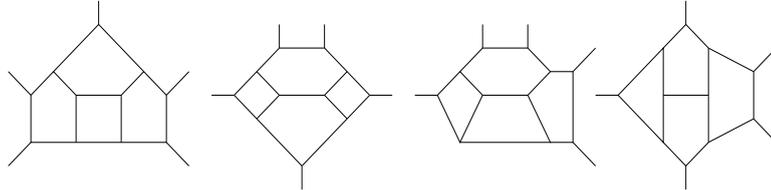
Figure 6.11: All graphs at four-loops with five external particles which could potentially give an dual-conformally invariant μ^2-term contribution.

6.9   CONCLUSIONS

Let us now briefly review the contents of this chapter and attempt to draw out conclusions and future directions for research. We set out the X-notation and attempted to take momentum-twistor expressions for 5- and 6-point loop amplitudes [9] and then rewrite these in traces of X's. Once we had achieved this we dimensionally regularised the integration variables $X_0, X_{0'}, \ldots$ to D=6-2ε dimensions but kept all terms inside the trace projected to 6-dimensions. We then demonstrated that we can recover the standard terms mapping back to momentum twistors in agreement with [6, 22, 102, 103]. In addition we compared our $\mathcal{O}(\varepsilon)$-terms to those we wanted to obtain [102] to see if we naturally obtained the correct terms from the momentum twistor integrands, which need only be valid before dimensional regularisation. We found that naively regularising the momentum-twistor integrands did not naturally produce the correct $\mathcal{O}(\varepsilon)$-terms.

We found that once one started to consider the dimensionally-regularised results of the amplitudes from [9] written in these variables then the correct $\mathcal{O}(\varepsilon)$-terms appeared only if the "reference twistor" was of the correct form. That is the five-point, one-loop equation with a single reference twistor only gave the correct dual-conformally invariant μ^2-term (i.e. 0) if X was a linear combination of the external points. If we set $X=I$ we could get a μ^2 contribution which we know does not occur. At 6-points this equation gives us

no parity-even contribution at $\mathcal{O}(\varepsilon)$ whatever we set the reference twistor X to be, so we looked at a different equation from [9] which had two reference twistors. This equation gave us some of the correct $\mathcal{O}(\varepsilon)$-terms if we set the reference twistors to be consecutive external points $X{=}X_i, Y{=}X_{i+1}$. As such we saw that it is indeed possible to obtain the correct $\mathcal{O}(\varepsilon)$-terms from this process but that the reference twistors are no longer arbitrary, certainly we would not have found the correct terms if we had set $X{=}Y{=}I$ the infinity twistor. Continuing to the six-point, two-loop amplitude we found that there was a term missing from the all orders in epsilon expansion after dimensional regularisation, as such we confirmed that this process will not generate the all-orders in ε-expansion.

During our comparison between five- and six-point one-loop and two-loop amplitudes we also came across an interesting and new way to collect several terms from [22, 102] together into a single object. From our form of writing the terms we naturally obtain some parity-odd terms of the form:

$$\frac{\epsilon(X_1, X_2, X_3, X_4, X_5, X_0)}{(X_1 \cdot X_0) \cdot s(X_5 \cdot X_0)} \qquad \frac{(X_1 \cdot X_3)\epsilon(X_1, X_2, X_3, X_4, X_5, X_0)}{(X_1 \cdot X_{0'}) \cdot s(X_3 \cdot X_{0'})(X_0 \cdot X_{0'})(X_0 \cdot X_3) \cdot s(X_0 \cdot X_1)} \tag{6.9.1}$$

where we draw these diagrams in Fig.6.12
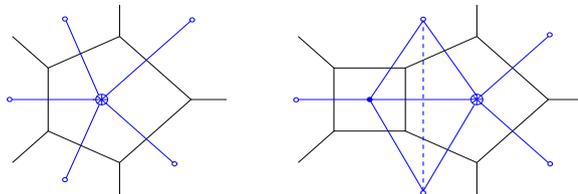
Figure 6.12: Parity-odd terms for five external particles at one- and two-loops which need to be divided into several pieces once we map back to momentum twistors and match results with [102] and [22].

For example the first of these terms - the five-point, one-loop pentagon - when mapped back into momentum twistors gives six terms Fig.6.13. When we expand $\varepsilon(X_1, X_2, X_3, X_4, X_5, \cdot){=}\sum_{i=1}^{5} \alpha_i X_i{+}\beta I$ which correspond when connected with X_0 to five one-mass boxes and a non dual-conformally invariant pentagon. We discussed the mechanics of this mapping in detail in (6.3.11) and the work following it, but here we wish to make a few remarks. There are many reasons why our terms seem to us to be much nicer objects than their corresponding momentum twistor collections. Firstly there is simply the issue of our single parity-odd object being broken into several parity-even terms.
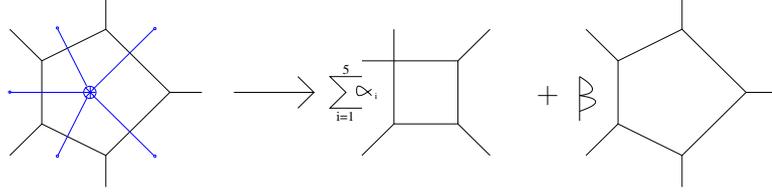
Figure 6.13: From a single parity-odd term written in the six-dimensional bi-twistor notation, mapping back into momentum twistors divides this single term into five one-mass boxes and a non dual-conformally invariant pentagon.

Alongside this there comes the introduction of the infinity-twistor and in turn a term which fails to be conformally invariant. These reasons alone would be a good argument for the X-notation to be considered an excellent tool for concisely expressing amplitude integrands.

Having explored one- and two-loops for five- and six-point amplitudes we presented a method for constructing an ansatz for $\mathcal{O}(\varepsilon)$ terms at higher loop-order or number of external particles. This can be seen at its most simple to involve drawing all planar-graphs with n external particles using only 3-valent vertices which had sufficient free integration variables in the numerator to contract together. We claim that this method means dual-conformally invariant μ^2-terms cannot appear until certain loop-orders depending on the number of external particles: 4-points $\leftrightarrow$ 4-loops, 5-points $\leftrightarrow$ 3-loops, $\geq$ 6-points $\leftrightarrow$ 1-loop. Indeed we can immediately write down the single allowed dual-conformally invariant μ^2-term for five-points at 3-loops.

$$\alpha \frac{\mu_{AB} \cdot \mu_{EF}(X_1 \cdot X_3)(X_1 \cdot X_4)}{(X_{AB} \cdot X_2) \cdots (X_{AB} \cdot X_4)(X_{AB} \cdot X_{CD})(X_{CD} \cdot X_{EF})}$$
$$\times \frac{(X_2 \cdot X_4)(X_2 \cdot X_5)}{(X_{CD} \cdot X_1)(X_{CD} \cdot X_2)(X_{EF} \cdot X_4) \cdots (X_{EF} \cdot X_1)} + \text{cyclic} \qquad (6.9.2)$$

Equally we saw there was only one-term at four-points and four-loops however the four-point MHV $\mathcal{N}{=}4$ SYM amplitudes have no dual-conformally invariant μ^2-terms through five-loops. Therefore understanding why this term does not in fact contribute is the most natural and important question which arises from this research.

One of the most severe limiting factors for taking this work further is the paucity of data of $\mathcal{O}(\varepsilon)$-results. Indeed beyond those $\mathcal{O}(\varepsilon)$-terms we explored here above, there are at the time of writing this and to the best of the author's knowledge no further terms known for any relevant amplitude. Potentially

knowing results for five-points beyond three-loops and six-points beyond two-loops would shed light on the questions and predictions of this work.

At present all amplitudes with ≥ 5 external particles are understood poorly and there are a very limited number of results as compared to the four-point case. As such potentially the most direct and fertile ground for continuing this investigation is to calculate the $\mathcal{O}(\varepsilon)$-integrand terms for the four-particle amplitudes. We have here written down the ansatz for the single $\mathcal{O}(\varepsilon)$ term which appears at four-loops, still at five-loops there are still a heavily limited number of graphs which can contribute a μ^2-term Fig.6.14. Understanding at
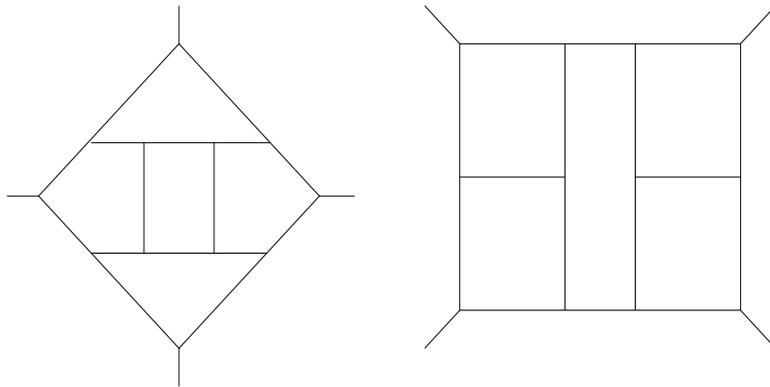
Figure 6.14: Graphs which can contribute at $\mathcal{O}(\varepsilon)$ at the level of the four-point, five-loop amplitude.

which loop-order these terms do begin to appear and the relations between their coefficients/numerators and those of the 'related' standard ($\mathcal{O}(\varepsilon^0)$) terms in the manner discussed earlier (Fig.6.7) may illuminate the as yet unresolved issues.

7

CONCLUSIONS

In this final section we will review the principal results of our earlier work at each stage and suggest the most pertinent potentials for future research. We note that at the end of earlier chapters we have already provided some concluding remarks regarding the work presented in that chapter. As such in this final section we will limit ourselves to discussing only the principal results and drawing out what seems to us to be the most likely avenue for future research.

In Chapter.3 we incorporated the R-invariants and as such the NMHV amplitudes into the two-dimensional kinematical picture joining the MHV amplitudes at two- and three-loops [89, 91]. We achieved this by finding a basis where finite-combinations can be written in a manifestly 1+1-dimensional fashion (3.3.2). Additionally in Chapter.2 we provided a more compact formula for the action of collinear limits acting linearly on the log of the amplitude. Especially in 2d kinematics we find remarkably simple formulae (2.8.5), later in Chapter.4 generalising this to find an equally simple formula even in the full helicity-changing limits (4.7.6).

Using these two developments together in Chapter.3 we used entirely symmetry arguments to be the first to construct the n-point one-loop NMHV amplitude without any additional restrictions on the kinematics [80]. In [46] through following our earlier suggestion and restricting the SU(4) R-symmetry to SU(2) × SU(2), our work at NMHV provides an improvement on this system as described in Chapter.3, in part as our form of expressing this amplitude made manifest the connection (proved through the $\bar{Q}$-equation) of the NMHV one-loop amplitude to the MHV 2-loop amplitude.

Using this similarity of form between different amplitudes at n-points, ℓ-loop N^kMHV and n-points $(\ell - 1)$-loop N^{k+1}MHV motivated our search for a universal method for constructing and uplifting these formulae to construct higher point amplitudes from their lower-point counterparts. We proposed a generalisation of the methods in [91] for the construction of higher-point, MHV, ℓ-loop amplitudes from lower-point, MHV, $(\ell - 1)$-loop amplitudes with an additional collinear vanishing part (4.3.1). Then taking this as a starting point we proposed an analogous formula for all n-point, N^kMHV, ℓ-loop amplitudes which has subsequently been used to construct N^2MHV one-loop and NMHV two-loop amplitudes [47].

We mentioned at the end of Chapter.4 several avenues for further research however we will repeat the most prominent here. Further calculations of more complicated amplitudes is possible, however these calculations will become increasingly computationally inefficient without further symmetries or other restrictions on the allowable components. One such potential restriction is the $\bar{Q}$-equation (3.5.6) [46] which can be seen to give restrictions which are additional to those appearing from cyclicity, parity, integrability etc. A compact form for seeing and implementing these restrictions would potentially hugely help in reducing the number and form of the component functions allowable. In addition it is possible that the OPE restrictions [3] which were used in [91] should continue to provide additional restrictions of more complicated amplitudes. Ultimately however, large steps forward in several areas, including the rewriting of integrable symbols into generalised logarithms etc., would need to be taken for this technique to be applicable to much more complicated amplitudes.

In Chapter.5 we extended work done using the correlator/amplitude duality at four-points [69] to incorporate complications arising at five-points [6]. Using this refined duality (5.4.5) we constructed the parity-even amplitude to six-loops and the parity-odd amplitude to five-loops [6]. The most natural next steps are to continue this process to higher-loop level or to attempt the same process for $n = 6$ (5.8.4). However both of these avenues have complications: for the $n = 5$ case the number of terms quickly grows unwieldy as we go to higher and higher loop-level, and there seems no obvious fix for this growth. In trying to go to 6-points we add a new complication that we now have N^2MHV $\times$ MHV and $(NMHV)^2$ contributions, not just MHV $\times \overline{\text{MHV}}$ terms we have had thus far. These extra amplitude contributions may be easy to isolate due to

their differing pole structure, however this process will necessarily be more complicated.

In Chapter.6 we rewrote amplitudes from momentum-twistor expressions [9] to six-dimensional X-notation and attempted to dimensionally regularise these expressions to $D = 4 - 2\varepsilon$-dimensions, matching with previously known results [22, 103]. We found that our expressions in X-space had several advantages over the previously utilised momentum-twistor expressions. In particular our single term

$$\frac{\epsilon(X_1, \cdots, X_5, x_0)}{(X_1 \cdot X_0) \cdots (X_5 \cdot X_0)} \tag{7.0.1}$$

becomes many terms in the map to momentum-twistors including losing explicit dual-conformal invariance. We also used our insights to construct an ansatz for the $\mathcal{O}(\varepsilon)$ terms at higher-loop level or higher-point amplitudes, giving these terms explicitly for five-point three-loops and four-point four- and five-loops. To continue our ansatz to five-point four-loops etc. would not be difficult however it is confirmation for these predictions in more complicated, dimensionally-regularised amplitudes e.g. four-points and five-loops which would be most useful for pushing this work onwards. Even if the predictions prove to be too large our work has severely limited the number of possible terms which can occur at $\mathcal{O}(\varepsilon)$ at least for four-loops and/or low-point amplitudes.

APPENDIX A

SYMBOLS AND FUNCTIONS AT 3-LOOPS

Let us first replicate the work presented in [80] before we present the additional work done in [47] and the terms as presented there. The conjecture at the centre of the method outlined in [91] for constructing MHV amplitudes in special kinematics states that (the logarithms of) the fundamental cross-ratios u_{ij} form the basis for the vector space on which the *symbol* of the amplitude is defined.

In the formalism of [91] 8-point, 3-loop, MHV amplitudes have the following structure:

$$\tilde{\mathcal{R}}_8^{(3)} = \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma^+(u_1) f_\tau^+(u_2) \tag{A.0.1}$$

where $a_{\sigma\tau} = a_{\tau\sigma}$ are rational coefficients, and the sum is over the set of functions f_σ^+ with the properties given in (4.1.4). The total polylog weight of $\tilde{\mathcal{R}}_8^{(3)}$ must be six which implies that the transcendental weights of individual functions f_σ^+ can be 2, 4 and 3. We can now similarly write down the expression for S_8,

$$
\begin{aligned}
S_8^{(3)}(x_2, x_4, x_6, x_8) &= \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma^+(u_1) f_\tau^+(u_2) + b_{\sigma\tau} f_\sigma^-(u_1) f_\tau^-(u_2) \\
&= \frac{1}{2}\tilde{\mathcal{R}}_8^{(3)} + T_8^{(3)}(x_2, x_4, x_6, x_8)
\end{aligned}
\tag{A.0.2}
$$

with $b_{\sigma\tau} = b_{\tau\sigma}$ and which utilize functions $f_\sigma^\pm$ with the property

$$f_{\sigma(u)}^\pm = \pm f_{\sigma(v)}^\pm \quad , \quad v = 1-u. \tag{A.0.3}$$

In [91] all possible (symbols and) functions $f_\sigma^+(u)$ were listed. It is straight-forward to generalise this construction to functions $f_\sigma^\pm$. For weight-2 there is only one function with properties (4.1.4) or (A.0.3),

$$\text{weight 2:} \quad f_{\text{weight 2}}^+(u) = \log(u)\log(v)\,. \tag{A.0.4}$$

This weight-2 function is accompanied in (A.0.1) by functions $f_\sigma^+(u)$ of weight-4. For completeness we list below symbols for all functions $f_\sigma^\pm(u)$. They come in two types, type-a and type-b:

$$\text{weight 4 a} \quad : \quad \begin{aligned} \text{Symb}[f_{a1}^\pm] &:= u \otimes u \otimes u \otimes v \pm v \otimes v \otimes v \otimes u \\ \text{Symb}[f_{a2}^\pm] &:= u \otimes u \otimes v \otimes u \pm v \otimes v \otimes u \otimes v \\ \text{Symb}[f_{a3}^\pm] &:= u \otimes v \otimes u \otimes u \pm v \otimes u \otimes v \otimes v \\ \text{Symb}[f_{a4}^\pm] &:= v \otimes u \otimes u \otimes u \pm u \otimes v \otimes v \otimes v \end{aligned} \tag{A.0.5}$$

$$\text{weight 4 b} \quad : \quad \begin{aligned} \text{Symb}[f_{b1}^\pm] &:= u \otimes u \otimes v \otimes v \pm v \otimes v \otimes u \otimes u \\ \text{Symb}[f_{b2}^\pm] &:= u \otimes v \otimes u \otimes v \pm v \otimes u \otimes v \otimes u \\ \text{Symb}[f_{b3}^\pm] &:= u \otimes v \otimes v \otimes u \pm v \otimes u \otimes u \otimes v \end{aligned} \tag{A.0.6}$$

It is important to note that as there is no function f^- of weight 2 with the desired properties and no mixed terms, i.e. $f_\sigma^+ f_\tau^-$, are possible in (A.0.2), there are no contributions to S_8 (or to T_8) from the weight-4 functions f^-.

What remains is to examine the weight-3 functions, known as type-c. Here we have (cf. [91]),

$$\text{weight 3 c} \quad : \quad \begin{aligned} \text{Symb}[f_{c1}^\pm] &:= u \otimes u \otimes v \pm v \otimes v \otimes u \\ \text{Symb}[f_{c2}^\pm] &:= u \otimes v \otimes u \pm v \otimes u \otimes v \\ \text{Symb}[f_{c3}^\pm] &:= u \otimes v \otimes v \pm v \otimes u \otimes u \end{aligned} \tag{A.0.7}$$

For the 8-point 3-loop amplitude itself, only the functions f^+ appear in Eq. (A.0.1). After imposing the constraint arising from the near-collinear OPE of [77] the final result of Ref. [91] for the octagon at 3-loops is given by

$$\begin{aligned} \tilde{\mathcal{R}}_8^{(3)} &= \log u_1 \log(1-u_1) \left[\alpha_1 \, f_{a3}^+(u_2) + \alpha_2 \, f_{a4}^+(u_2) + \alpha_3 \, f_{b2}^+(u_2) + \alpha_4 \, f_{b3}^+(u_2) \right] \\ &\quad + \alpha_5 f_{c2}^+(u_1) f_{c2}(u_2) + \alpha_6 f_{c2}^+(u_1) f_{c3}^+(u_2) + \alpha_7 f_{c3}^+(u_1) f_{c3}^+(u_2) \\ &\quad + f_{c1}^+(u_1) \left[\frac{1}{2} f_{c1}^+(u_2) + 2 f_{c2}^+(u_2) + f_{c3}^+(u_2) \right] \\ &\quad + (u_1 \leftrightarrow u_2) \end{aligned} \tag{A.0.8}$$

with the f_a^+, f_b^+ and f_c^+ functions are straightforwardly reconstructed from their symbols in (A.0.5)-(A.0.7) and are listed in Eqs. (5.15) of Ref. [91].

To fully determine S_8 at 3 loops, in addition to $\tilde{\mathcal{R}}_8^{(3)}$ we need the contribution $T_8^{(3)}$ in (A.0.2) which comes solely from the f^- functions. Since there are no f^- contributions at weight 2, the contributions to $T_8^{(3)}$ can arise only from the weight-3 times weight-3 functions f^- in (A.0.7). This gives 6 possible functions

$$T_8^{(3)} \ni \begin{array}{l} f_{c1}^-(u_1,u_3)f_{c1}^-(u_2,u_4) \\ f_{c1}^-(u_1,u_3)f_{c2}^-(u_2,u_4){+}f_{c2}^-(u_1,u_3)f_{c1}^-(u_2,u_4) \\ f_{c1}^-(u_1,u_3)f_{c3}^-(u_2,u_4){+}f_{c3}^-(u_1,u_3)f_{c1}^-(u_2,u_4) \\ f_{c2}^-(u_1,u_3)f_{c2}^-(u_2,u_4) \\ f_{c2}^-(u_1,u_3)f_{c3}^-(u_2,u_4){+}f_{c3}^-(u_1,u_3)f_{c2}^-(u_2,u_4) \\ f_{c3}^-(u_1,u_3)f_{c3}^-(u_2,u_4) \end{array} \tag{A.0.9}$$

In [47] the authors dropped the assumption which had been used up until that point that the symbol only had entries which were simple cross-ratios. It was found at the level of NMHV 2-loops and MHV 3-loops that the basis for the symbol must include terms of the form $u_1 - u_2$ which is cyclically related to $1{-}u_1{-}u_2$. These entries cannot appear arbitrarily in the symbol as they cannot be found in the first or last place of the symbol tensor. It is also only such simple linear combinations which can appear, however even this small addition to the possible entries for the symbol adds a large amount of possible terms for more complex amplitudes. Here we will only include the amplitude as presented in [47] and leave the derivation and exploration to be found there for the interested reader.

Here the cross-ratios are defined differently with $v = \frac{\langle 13\rangle\langle 57\rangle}{\langle 17\rangle\langle 35\rangle}$ and $w = \frac{\langle 24\rangle\langle 68\rangle}{\langle 28\rangle\langle 46\rangle}$. Using these the 8-point amplitude is given as

$$R_{8,0}^{(3)} = \left[(f_{8,0}^{a(3)}(v,w){+}(v \leftrightarrow \tfrac{1}{v})){+}(w \leftrightarrow \tfrac{1}{w}){+}f_{8,0}^{b(3)}(v,w) \right] {+}(v \leftrightarrow w){+}f_{8,0}^{c(3)}(v,w) \tag{A.0.10}$$

where the "non-trivial" part is contained solely in $f_{8,0}^{a(3)}$:

$$\begin{aligned} f_{8,0}^{a(3)}(x,y) ={}& 2\mathrm{Li}_{2,2,2}(\tfrac{1}{1+x},1{+}x,\tfrac{1}{1+y}){+}2\mathrm{Li}_{1,2,2}(\tfrac{1}{1+x},1{+}x,\tfrac{1}{1+y})\log(1{+}x){+}\mathrm{Li}_{2,4}(-\tfrac{1}{x},xy) \\ &+\mathrm{Li}_{1,4}(-\tfrac{1}{x},xy)\log(x){-}\mathrm{Li}_{2,2,2}(\tfrac{1}{1+x},1{+}x,1){-}\mathrm{Li}_{2,2,2}(1,1,\tfrac{1}{1+y}) \\ &-\mathrm{Li}_{1,2,2}(\tfrac{1}{1+x},1{+}x,1)\log(1{+}x) \end{aligned} \tag{A.0.11}$$

The remaining functions involve only classical or lower-weight polylogs, with various symmetry properties:

$$f_{8,0}^{b(3)}(x,y) = -2\log\frac{(1+y)^2}{y}\left[5\text{Li}_5(-x) - \text{Li}_4(-x)\log(x) + 7\zeta_4\log(1+x) + \frac{1}{6}\zeta(2)\log^3 x + \frac{1}{4}\zeta_2\log\frac{(1+x)^2}{x}\log^2 x\right]$$
$$- \left[2\text{Li}_{3,1}(-x,1) + 2\text{Li}_{2,2}(1,-x) + 2\text{Li}_4(-x) + 6\text{Li}_{3,1}(1,-x) - 2\log x\text{Li}_{2,1}(1,-x) + 4\zeta_3\log(1+x)\right]$$
$$\times \log(1+y)\log\frac{1+y}{y} + \left(\frac{1}{24}\log^4 x - \frac{31}{4}\zeta_4\right)\left(\frac{1}{2}\log^2 y - \zeta_2\right) \tag{A.0.12}$$

$$f_{8,0}^{c(3)}(x,y) = 2\left[2\text{Li}_3(-x) - \text{Li}_2(-x)\log x - \frac{1}{2}\log^2(x)\log(1+x) + \frac{1}{6}\log^3 x - \zeta_2\log\frac{1+x}{x}\right] \times (x \leftrightarrow y) - \frac{67\pi^6}{1260}$$
$$- \log\frac{(1+y)^2}{y}\log\frac{(1+x)^2}{x}\left[\frac{2}{3}\log(1+y)\log(1+\frac{1}{y})\log(1+x)\log(1+\frac{1}{x}) + \frac{1}{12}\log^2 x\log^2 y - \frac{9}{2}\zeta_4\right] \tag{A.0.13}$$

Then proceeding to try and write this in an S-formula we define

$$S_{8,0}^{(3)} = \left[f_{8,0}^{a(3)}(v,w) + f_{8,0}^{a(3)}(\frac{1}{v},\frac{1}{w}) + \frac{1}{2}f_{8,0}^{b(3)}(v,w)\right] + (v \leftrightarrow w) + \frac{1}{2}f_{8,0}^{c(3)}(v,w) + \delta_{8,0}^{(3)}(v,w) \tag{A.0.14}$$

where $\delta_{8,0}^{(3)}(v,w)$ is an undetermined function with the symmetry properties $\delta_{8,0}^{(3)}(v,w) = \delta_{8,0}^{(3)}(w,v) = -\delta_{8,0}^{(3)}(1/v,w)$. It can be chosen such that $S_{8,0}^{(3)}$ does not diverge in on-shell limits, and by computing the 10-point symbol the authors found that it could also be chosen such that only $\log^6(\cdot)$ and $\text{Li}_2(\cdot)\log^4(\cdot)$ terms remain at 10-points after subtracting the uplifting. Furthermore, all terms of the symbol of the remainder turn out to have three odd and three even twistor entries. However, since the 10-point MHV amplitude is as yet insufficient to fix the complete n-point amplitude, the authors did not attach a full formula for $S_{8,0}^{(3)}$ but instead stated the general conjecture and left the further computations open. Note that in these calculations the assumptions made in [80, 91] were modified in the following ways (as already mentioned in the main text)

- For octagons in 2d kinematics to all loops, only 6 "letters" can appear in the symbol:$v, w, 1+v, 1+w, v-w$ and $1-vw$, all of which are already seen at 3-loops.

- The last entry of the symbol for MHV and NMHV octagons can only be $v, w, 1+v$ or $1+w$.

These slight relaxations of earlier assumptions are necessary and although it complicates the calculation, it does not cause all calculations to become calculationally impractical.

We now turn our attention to the 10-point amplitude, which was originally obtained in [91] in the form given by Eq. (4.1.7) and was not modified by work done in [47]. The first term on the right-hand side gives a particular solution to the multi-collinear constraints. It is reproduced by the S_8 contributions (specifically by the $f^+ f^+$ terms in (A.0.2). On the other hand, the second term, V_{10} denotes a generic 10-point function which is constrained to vanish in all triple collinear limits. This collinearly vanishing contribution was constructed in [91].

Here, for convenience of the reader, we reproduce the form of V_{10} from [91]. In order to be able to uplift the 10-point result to 12 points and all higher points using our general S-formula we do not need S_{12} but we need to know that it can be deconstructed in terms of collinearly vanishing T_8 and collinearly vanishing S_{10} contributions.

At 10-points there are 10 fundamental cross-ratios

$$u_i := u_{i,i+4}\,, \qquad i = 1, \dots, 10 \tag{A.0.15}$$

which can be divided into 5 parity-even $(u_1, u_3, \dots, u_9)$, and 5 parity-odd cross-ratios $(u_2, u_4, \dots, u_{10})$. It was argued in [91] that V_{10} is assembled from functions of parity-even cross-ratios times functions of parity-odd u's as follows:

$$f_i(u_{\text{even}}) f_j(u_{\text{odd}}) + \text{cyclic} + \text{parity}\,, \tag{A.0.16}$$

where these functions f_i must themselves vanish in any collinear limit. To do this they must have weight-3 and each term must contain 3 consecutive cross-ratios of given parity, e.g. u_1, u_3, u_5. They are not difficult to find analytically [91]:

$$f_1(u_1, u_3, u_5) = \log(u_1) \log(u_3) \log(u_5)$$
$$f_2(u_1, u_3, u_5) = \log(u_3) \left(\text{Li}_2(u_1) - \text{Li}_2(1-u_3) + \text{Li}_2(u_5) - \frac{\pi^2}{6} \right)$$
$$f_3(u_1, u_3, u_5, u_7, u_9) = \sum_{i=1,3,5,7,9} (\text{Li}_3(u_i) - \text{Li}_3(1-u_i)) - \zeta_3\,. \tag{A.0.17}$$

Here f_1 and f_2 give 5 independent functions via cyclic permutations of the arguments, whereas f_3 is cyclically symmetric giving only 1 independent function, as such we have 11 functions in total. These functions are combined together to give a total of 12 independent weight-6 collinear vanishing contributions to

V_{10}:

$$f_1(u_1, u_3, u_5)f_1(u_2, u_4, u_6) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_1(u_4, u_6, u_8) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_1(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_2(u_2, u_4, u_6) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_2(u_4, u_6, u_8) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_2(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity}$$
$$f_2(u_1, u_3, u_5)f_2(u_2, u_4, u_6) + \text{cyclic} + \text{parity}$$
$$f_2(u_1, u_3, u_5)f_2(u_4, u_6, u_8) + \text{cyclic} + \text{parity}$$
$$f_2(u_1, u_3, u_5)f_2(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity}$$
$$f_1(u_1, u_3, u_5)f_3(u_i^-) + \text{cyclic} + \text{parity}$$
$$f_2(u_1, u_3, u_5)f_3(u_i^-) + \text{cyclic} + \text{parity}$$
$$f_3(u_1, u_3, u_5)f_3(u_i^-) + \text{cyclic} + \text{parity} \tag{A.0.18}$$

Appendix B

Four-Point, Six-Loop, Planar Topologies

Here we reproduce from [69] the 23 numerator polynomials for the four-point, six-loop f-graphs of the rung-rule type:

$$P_1^{(6)} = \frac{1}{2}x_{13}^4 x_{18}^2 x_{19}^2 x_{24}^2 x_{27}^2 x_{28}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{46}^4 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_2^{(6)} = \frac{1}{2}x_{13}^4 x_{18}^4 x_{24}^2 x_{27}^2 x_{28}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{46}^4 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^4 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_3^{(6)} = \frac{1}{8}x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^2 x_{28}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_4^{(6)} = \frac{1}{2}x_{13}^2 x_{16}^4 x_{18}^2 x_{24}^4 x_{27}^2 x_{28}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{49}^6 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 + \dots,$$

$$P_5^{(6)} = \frac{1}{2}x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^6 x_{27}^2 x_{36}^2 x_{38}^2 x_{39}^2 x_{48}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 + \dots,$$

$$P_6^{(6)} = \frac{1}{4}x_{13}^2 x_{17}^4 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{36}^2 x_{38}^4 x_{39}^2 x_{46}^2 x_{47}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{78}^2 x_{1,10}^2 x_{2,10}^2 x_{5,10}^2 x_{6,10}^4 + \dots,$$

$$P_7^{(6)} = \frac{1}{2}x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{27}^2 x_{36}^4 x_{39}^2 x_{48}^4 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 + \dots,$$

$$P_8^{(6)} = \frac{1}{8}x_{13}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{24}^4 x_{27}^2 x_{28}^2 x_{39}^6 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^4 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_9^{(6)} = \frac{1}{2}x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{69}^2 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_{10}^{(6)} = \frac{1}{2}x_{13}^2 x_{16}^2 x_{17}^2 x_{24}^6 x_{29}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{69}^2 x_{79}^2 x_{1,10}^4 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 + \dots,$$

$$P_{11}^{(6)} = x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 + \dots,$$

$$P_{12}^{(6)} = \frac{1}{4}x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{79}^4 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_{13}^{(6)} = \frac{1}{16}x_{13}^4 x_{16}^2 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \dots,$$

$$P_{14}^{(6)} = \frac{1}{4}x_{13}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{24}^8 x_{35}^2 x_{38}^2 x_{39}^2 x_{48}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^4 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 + \dots,$$

$$P_{15}^{(6)} = \frac{1}{2}x_{13}^2 x_{16}^2 x_{18}^4 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^4 x_{39}^4 x_{46}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{1,10}^2 x_{2,10}^2 x_{5,10}^2 x_{6,10}^2 x_{7,10}^2 + \dots,$$

$$P_{16}^{(6)} = \frac{1}{4} x_{13}^4 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{78}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 x_{7,10}^2 + \cdots,$$

$$P_{17}^{(6)} = \frac{1}{4} x_{13}^2 x_{15}^2 x_{17}^2 x_{18}^2 x_{24}^6 x_{27}^2 x_{36}^4 x_{39}^4 x_{45}^2 x_{48}^2 x_{58}^2 x_{59}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{1,10}^2 x_{2,10}^2 x_{5,10}^2 x_{6,10}^2 x_{7,10}^2 + \cdots,$$

$$P_{18}^{(6)} = \frac{1}{2} x_{13}^2 x_{17}^4 x_{19}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{36}^4 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{78}^2 x_{89}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{4,10}^2 x_{5,10}^2 + \cdots,$$

$$P_{19}^{(6)} = \frac{1}{4} x_{13}^2 x_{16}^4 x_{19}^2 x_{24}^6 x_{28}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{89}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{7,10}^2 + \cdots,$$

$$P_{20}^{(6)} = \frac{1}{32} x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^{10} x_{36}^2 x_{38}^2 x_{39}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{89}^2 x_{1,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 x_{7,10}^2 + \cdots,$$

$$P_{21}^{(6)} = x_{13}^2 x_{17}^4 x_{19}^2 x_{24}^4 x_{26}^4 x_{36}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{48}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{89}^2 x_{1,10}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{8,10}^2 + \cdots,$$

$$P_{22}^{(6)} = \frac{1}{2} x_{13}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{37}^4 x_{39}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{89}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 x_{8,10}^2 + \cdots,$$

$$P_{23}^{(6)} = \frac{1}{2} x_{13}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{28}^4 x_{37}^2 x_{38}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{2,10}^2 x_{3,10}^2 x_{5,10}^2 x_{6,10}^2 x_{9,10}^2 + \cdots,$$

$$\text{(B.0.1)}$$

where the ellipses denote all the terms obtained through application of the S_{10}-permutation of indices.

There are 13 additional polynomials not fixed by the rung-rule method of [69]:

$$P_{24}^{(6)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{37}^2 x_{45}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{59}^4 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{2,10}^2 x_{6,10}^2 x_{7,10}^2 x_{8,10}^2 + \cdots,$$

$$P_{25}^{(6)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{37}^2 x_{45}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{59}^4 x_{67}^2 x_{78}^4 x_{89}^2 x_{2,10}^4 x_{6,10}^2 x_{7,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{26}^{(6)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{37}^2 x_{45}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{2,10}^4 x_{6,10}^2 x_{7,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{27}^{(6)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{27}^2 x_{29}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{89}^4 x_{1,10}^4 x_{5,10}^2 x_{7,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{28}^{(6)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{27}^2 x_{29}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{1,10}^4 x_{5,10}^2 x_{8,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{29}^{(6)} = \frac{1}{20} x_{12}^2 x_{13}^2 x_{16}^2 x_{19}^2 x_{23}^2 x_{24}^2 x_{27}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{1,10}^2 x_{2,10}^2 x_{5,10}^2 x_{8,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{30}^{(6)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{15}^2 x_{17}^2 x_{23}^2 x_{24}^2 x_{27}^2 x_{29}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{46}^2 x_{56}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{1,10}^2 x_{4,10}^2 x_{6,10}^2 x_{8,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{31}^{(6)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{15}^2 x_{17}^2 x_{19}^2 x_{23}^2 x_{24}^2 x_{27}^2 x_{28}^2 x_{34}^2 x_{35}^2 x_{45}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{3,10}^2 x_{6,10}^4 x_{8,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{32}^{(6)} = \frac{1}{16} x_{12}^2 x_{13}^2 x_{14}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{29}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{47}^2 x_{56}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{1,10}^2 x_{5,10}^2 x_{7,10}^2 x_{8,10}^2 x_{9,10}^2 + \cdots,$$

$$P_{33}^{(6)} = \frac{1}{16} x_{12}^2 x_{13}^2 x_{14}^2 x_{16}^{23} x_{18}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{28}^2 x_{34}^2 x_{35}^2 x_{37}^2 x_{45}^2 x_{47}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2 x_{5,10}^2 x_{6,10}^2 x_{9,10}^6 + \cdots,$$

$$P_{34}^{(6)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{14}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{29}^2 x_{34}^2 x_{35}^2 x_{38}^2 x_{45}^2 x_{47}^2 x_{56}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{89}^4 x_{1,10}^2 x_{5,10}^2 x_{7,10}^4 x_{9,10}^2 + \cdots,$$

$$P_{35}^{(6)} = \frac{1}{8} x_{12}^2 x_{13}^2 x_{15}^2 x_{16}^2 x_{23}^2 x_{24}^2 x_{28}^4 x_{34}^2 x_{37}^4 x_{45}^2 x_{46}^2 x_{49}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{1,10}^2 x_{5,10}^2 x_{9,10}^6 + \cdots,$$

$$P_{36}^{(6)} = \frac{1}{8} x_{12}^2 x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{23}^2 x_{25}^4 x_{34}^4 x_{37}^2 x_{45}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{78}^4 x_{89}^2 x_{2,10}^2 x_{6,10}^2 x_{7,10}^2 x_{9,10}^4 + \cdots,$$

$$\text{(B.0.2)}$$

APPENDIX C

FOUR-POINT, FOUR-LOOP TOPOLO-GIES

We give the 29 non-planar f-graphs of genus 1. We will not display them graphically but simply present the corresponding S_8-polynomials:

$$P_4^{(4)} = \frac{1}{16} x_{12}^2 x_{13}^2 x_{18}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 + \ldots,$$

$$P_5^{(4)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{18}^2 x_{23}^2 x_{26}^2 x_{34}^2 x_{45}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{78}^2 + \ldots,$$

$$P_6^{(4)} = \frac{1}{12} x_{12}^2 x_{13}^2 x_{18}^2 x_{23}^2 x_{26}^2 x_{34}^2 x_{45}^2 x_{47}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{78}^2 + \ldots,$$

$$P_7^{(4)} = \frac{1}{48} x_{12}^2 x_{16}^2 x_{18}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{38}^2 x_{45}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{78}^2 + \ldots,$$

$$P_8^{(4)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{57}^2 x_{68}^2 x_{78}^4 + \ldots,$$

$$P_9^{(4)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{27}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{78}^4 + \ldots,$$

$$P_{10}^{(4)} = \frac{1}{8} x_{12}^2 x_{14}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{36}^2 x_{45}^2 x_{57}^2 x_{68}^2 x_{78}^4 + \ldots,$$

$$P_{11}^{(4)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{35}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{68}^2 x_{78}^2 + \ldots,$$

$$P_{12}^{(4)} = \frac{1}{4} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{35}^2 x_{48}^2 x_{56}^4 x_{67}^2 x_{78}^4 + \ldots,$$

$$P_{13}^{(4)} = \frac{1}{2} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{27}^2 x_{35}^2 x_{46}^2 x_{48}^2 x_{56}^4 x_{78}^4 + \ldots,$$

$$P_{14}^{(4)} = \frac{1}{4} x_{12}^4 x_{13}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{68}^2 x_{78}^4 + \ldots,$$

$$P_{15}^{(4)} = \frac{1}{2} x_{12}^4 x_{13}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{48}^2 x_{56}^4 x_{67}^2 x_{78}^4 + \ldots,$$

$$P_{16}^{(4)} = \frac{1}{12} x_{12}^4 x_{13}^2 x_{24}^2 x_{35}^2 x_{37}^2 x_{46}^2 x_{48}^2 x_{56}^4 x_{78}^4 + \ldots,$$

$$P_{17}^{(4)} = \frac{1}{8} x_{12}^4 x_{13}^2 x_{25}^2 x_{34}^4 x_{48}^2 x_{56}^4 x_{67}^2 x_{78}^4 + \ldots,$$

$$P_{18}^{(4)} = \frac{1}{8} x_{12}^4 x_{15}^2 x_{25}^2 x_{34}^4 x_{36}^2 x_{46}^2 x_{57}^2 x_{68}^2 x_{78}^4 + \dots,$$

$$P_{19}^{(4)} = \frac{1}{4} x_{12}^4 x_{15}^2 x_{25}^2 x_{34}^2 x_{36}^2 x_{37}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{78}^4 + \dots,$$

$$P_{20}^{(4)} = \frac{1}{4} x_{12}^4 x_{15}^2 x_{28}^2 x_{34}^4 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{78}^4 + \dots,$$

$$P_{21}^{(4)} = \frac{1}{8} x_{12}^4 x_{13}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2 + \dots,$$

$$P_{22}^{(4)} = \frac{1}{144} x_{12}^2 x_{14}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{36}^2 x_{45}^2 x_{56}^2 x_{78}^6 + \dots,$$

$$P_{23}^{(4)} = \frac{1}{8} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{35}^2 x_{46}^4 x_{56}^4 x_{78}^6 + \dots,$$

$$P_{24}^{(4)} = \frac{1}{8} x_{12}^4 x_{13}^2 x_{24}^2 x_{34}^4 x_{35}^2 x_{46}^2 x_{56}^4 x_{78}^6 + \dots,$$

$$P_{25}^{(4)} = \frac{1}{12} x_{12}^4 x_{13}^2 x_{25}^2 x_{34}^4 x_{46}^2 x_{56}^4 x_{78}^6 + \dots,$$

$$P_{26}^{(4)} = \frac{1}{16} x_{12}^4 x_{15}^2 x_{25}^2 x_{34}^4 x_{36}^2 x_{46}^2 x_{56}^2 x_{78}^6 + \dots,$$

$$P_{27}^{(4)} = \frac{1}{1152} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2 + \dots,$$

$$P_{28}^{(4)} = \frac{1}{96} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{56}^4 x_{57}^2 x_{68}^2 x_{78}^4 + \dots,$$

$$P_{29}^{(4)} = \frac{1}{32} x_{12}^4 x_{13}^2 x_{24}^2 x_{34}^4 x_{56}^4 x_{57}^2 x_{68}^2 x_{78}^4 + \dots,$$

$$P_{30}^{(4)} = \frac{1}{192} x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 x_{56}^2 x_{78}^6 + \dots,$$

$$P_{31}^{(4)} = \frac{1}{32} x_{12}^4 x_{13}^2 x_{24}^2 x_{34}^4 x_{56}^6 x_{78}^6 + \dots,$$

$$P_{32}^{(4)} = \frac{1}{384} x_{12}^6 x_{34}^6 x_{56}^6 x_{78}^6 + \dots,$$

$$(\text{C.0.1})$$

where the ellipses denote the terms with all possible S_8-permutation of indices.

Appendix D

Five-Point, Four-Loop Amplitudes

We continue the process used in Chapter.5 as done in [69] to obtain the full (parity-even and parity-odd part) four-loop, five-point amplitude and check that it satisfies the consistency condition (5.7.3). For the four-loop result

$$M_5^{(4)} = \frac{1}{2}\frac{1}{4!}\int d^4x_6 d^4x_7 d^4x_8 d^4x_9 \left(\sum_{i=1}^{71} c_i \mathcal{I}_i^{(4)}\right) \qquad (D.0.1)$$

where

$$c_1 = \cdots = c_{28} = c_{45} = \cdots = c_{62} = 1$$
$$c_{29} = \cdots = c_{44} = c_{63} = \cdots = c_{71} = -1$$

and

$$\mathcal{I}_1^{(4)} = \left[\frac{x_{13}^8 x_{25}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{57}^2 x_{68}^2 x_{69}^2 x_{78}^2}\right] \qquad \mathcal{I}_2^{(4)} = \left[\frac{x_{16}^6 x_{24}^2 x_{25}^2 x_{35}^2}{x_{17}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2}\right]$$

$$\mathcal{I}_3^{(4)} = \left[\frac{x_{13}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{19}^2 x_{28}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{56}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2}\right] \qquad \mathcal{I}_4^{(4)} = \left[\frac{x_{14}^2 x_{24}^2 x_{25}^2}{x_{17}^2 x_{26}^2 x_{29}^2 x_{48}^2 x_{49}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{89}^2}\right]$$

$$\mathcal{I}_5^{(4)} = \left[\frac{x_{13}^6 x_{25}^2 x_{35}^2 x_{46}^2}{x_{16}^2 x_{17}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{48}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{79}^2}\right] \qquad \mathcal{I}_6^{(4)} = \left[\frac{x_{13}^6 x_{24}^2 x_{67}^2}{x_{16}^2 x_{17}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2}\right]$$

$$\mathcal{I}_7^{(4)} = \left[\frac{x_{14}^6 x_{35}^2 x_{36}^2}{x_{16}^2 x_{17}^2 x_{19}^2 x_{38}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{89}^2}\right] \qquad \mathcal{I}_8^{(4)} = \left[\frac{x_{14}^2 x_{16}^4 x_{24}^2 x_{25}^2 x_{37}^2}{x_{17}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{46}^2 x_{47}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{89}^2}\right]$$

$$\mathcal{I}_9^{(4)} = \left[\frac{x_{16}^4 x_{24}^2 x_{25}^2 x_{35}^4}{x_{17}^2 x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^2}\right] \qquad \mathcal{I}_{10}^{(4)} = \left[\frac{x_{16}^4 x_{24}^2 x_{25}^2 x_{35}^2 x_{57}^2}{x_{17}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2}\right]$$

$$\mathcal{I}_{11}^{(4)} = \left[\frac{x_{13}^6 x_{14}^2 x_{27}^2 x_{35}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{57}^2 x_{69}^2 x_{78}^2 x_{89}^2}\right] \qquad \mathcal{I}_{12}^{(4)} = \left[\frac{x_{13}^2 x_{17}^4 x_{24}^4}{x_{16}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2}\right]$$

$$\mathcal{I}_{13}^{(4)} = \left[\frac{x_{16}^4 x_{24}^2 x_{25}^2 x_{35}^2 x_{37}^2}{x_{17}^2 x_{18}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2}\right] \qquad \mathcal{I}_{14}^{(4)} = \left[\frac{x_{13}^4 x_{17}^2 x_{25}^2 x_{35}^2 x_{46}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2}\right]$$

$$\mathcal{I}_{15}^{(4)} = \left[\frac{x_{13}^4 x_{17}^2 x_{24}^2 x_{46}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2}\right] \qquad \mathcal{I}_{16}^{(4)} = \left[\frac{x_{13}^2 x_{14}^4 x_{24}^2 x_{56}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2}\right]$$

$$\mathcal{I}_{17}^{(4)} = \left[\frac{x_{16}^2 x_{18}^2 x_{24}^2 x_{25}^2 x_{35}^2 x_{37}^2}{x_{17}^2 x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{48}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2}\right] \qquad \mathcal{I}_{18}^{(4)} = \left[\frac{x_{16}^2 x_{17}^2 x_{24}^2 x_{25}^4 x_{35}^2}{x_{18}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2}\right]$$

$$\mathcal{I}_{19}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{26}^2 x_{35}^4 x_{46}^2}{x_{16}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{20}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{26}^2 x_{35}^2 x_{36}^2 x_{47}^2}{x_{16}^2 x_{17}^2 x_{27}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{21}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{24}^2 x_{26}^2 x_{36}^2 x_{57}^2}{x_{16}^2 x_{19}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{22}^{(4)} = \left[\frac{x_{14}^2 x_{24}^4 x_{25}^2 x_{36}^2 x_{67}^2}{x_{16}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{23}^{(4)} = \left[\frac{x_{17}^2 x_{24}^2 x_{25}^4 x_{35}^2 x_{36}^2}{x_{19}^2 x_{26}^2 x_{28}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{24}^{(4)} = \left[\frac{x_{14}^4 x_{24}^2 x_{25}^2 x_{27}^2 x_{36}^2}{x_{17}^2 x_{19}^2 x_{26}^2 x_{28}^2 x_{29}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{25}^{(4)} = \left[\frac{x_{13}^2 x_{18}^2 x_{24}^4 x_{36}^2}{x_{16}^2 x_{17}^2 x_{26}^2 x_{29}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{26}^{(4)} = \left[\frac{x_{13}^2 x_{18}^2 x_{25}^2 x_{27}^2 x_{35}^2 x_{46}^2}{x_{17}^2 x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{48}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{27}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{19}^2 x_{28}^2 x_{35}^2 x_{46}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{26}^2 x_{36}^2 x_{39}^2 x_{48}^2 x_{49}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{28}^{(4)} = \left[\frac{x_{18}^2 x_{24}^4 x_{25}^2 x_{35}^2 x_{67}^2}{x_{16}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{58}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{29}^{(4)} = \left[\frac{x_{13}^4 x_{25}^2 x_{35}^2}{x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{30}^{(4)} = \left[\frac{x_{14}^4 x_{35}^2}{x_{18}^2 x_{19}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{31}^{(4)} = \left[\frac{x_{16}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{19}^2 x_{26}^2 x_{29}^2 x_{36}^2 x_{48}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{32}^{(4)} = \left[\frac{x_{14}^2 x_{16}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{17}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{48}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{33}^{(4)} = \left[\frac{x_{13}^4 x_{14}^2 x_{24}^2 x_{35}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{58}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{34}^{(4)} = \left[\frac{x_{14}^2 x_{17}^2 x_{24}^2 x_{25}^2}{x_{16}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{47}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{35}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{17}^2 x_{35}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{36}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{17}^2 x_{25}^2 x_{35}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{37}^{(4)} = \left[\frac{x_{16}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{19}^2 x_{28}^2 x_{29}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{38}^{(4)} = \left[\frac{x_{14}^2 x_{17}^2 x_{24}^2 x_{25}^2 x_{35}^2}{x_{16}^2 x_{18}^2 x_{28}^2 x_{29}^2 x_{39}^2 x_{47}^2 x_{49}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{39}^{(4)} = \left[\frac{x_{13}^4 x_{25}^4}{x_{16}^2 x_{19}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{56}^2 x_{58}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{40}^{(4)} = \left[\frac{x_{13}^4 x_{14}^2 x_{27}^2 x_{35}^2}{x_{16}^2 x_{17}^2 x_{18}^2 x_{26}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{49}^2 x_{58}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{41}^{(4)} = \left[\frac{x_{16}^2 x_{24}^4 x_{25}^2 x_{35}^2}{x_{19}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{46}^2 x_{48}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{42}^{(4)} = \left[\frac{x_{13}^2 x_{16}^2 x_{25}^2 x_{28}^2 x_{35}^2}{x_{17}^2 x_{18}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{39}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{43}^{(4)} = \left[\frac{x_{14}^2 x_{16}^2 x_{24}^2 x_{25}^2 x_{37}^2}{x_{17}^2 x_{18}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{49}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{44}^{(4)} = \left[\frac{x_{13}^2 x_{14}^2 x_{17}^2 x_{24}^2 x_{35}^2 x_{69}^2}{x_{16}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{45}^{(4)} = \left[\frac{i x_{13}^6 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{46}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{46}^{(4)} = \left[\frac{i x_{13}^2 \varepsilon_{123456}}{x_{16}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{38}^2 x_{47}^2 x_{57}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{47}^{(4)} = \left[\frac{i x_{13}^4 x_{14}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{48}^{(4)} = \left[\frac{i x_{16}^4 x_{25}^2 \varepsilon_{123456}}{x_{17}^2 x_{18}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{49}^{(4)} = \left[\frac{i x_{13}^4 x_{26}^2 \varepsilon_{123457}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{28}^2 x_{29}^2 x_{36}^2 x_{37}^2 x_{39}^2 x_{47}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{50}^{(4)} = \left[\frac{i x_{13}^4 x_{46}^2 \varepsilon_{123457}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{51}^{(4)} = \left[\frac{i x_{17}^4 x_{24}^2 \varepsilon_{123458}}{x_{16}^2 x_{18}^2 x_{19}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{52}^{(4)} = \left[\frac{i x_{13}^2 x_{14}^2 x_{37}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{53}^{(4)} = \left[\frac{i x_{14}^2 x_{17}^2 x_{28}^2 \varepsilon_{123457}}{x_{16}^2 x_{18}^2 x_{19}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{54}^{(4)} = \left[\frac{i x_{13}^2 x_{17}^2 x_{28}^2 \varepsilon_{123458}}{x_{16}^2 x_{18}^2 x_{19}^2 x_{27}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{55}^{(4)} = \left[\frac{i x_{13}^2 x_{14}^2 x_{27}^2 \varepsilon_{123457}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{28}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{56}^{(4)} = \left[\frac{i x_{13}^2 x_{46}^2 x_{56}^2 \varepsilon_{123457}}{x_{16}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{38}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{57}^{(4)} = \left[\frac{i x_{17}^2 x_{24}^2 x_{36}^2 \varepsilon_{123456}}{x_{16}^2 x_{18}^2 x_{27}^2 x_{28}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{58}^{(4)} = \left[\frac{i x_{14}^2 x_{18}^2 x_{37}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{38}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{59}^{(4)} = \left[\frac{i x_{13}^2 x_{14}^2 x_{67}^2 \varepsilon_{123459}}{x_{16}^2 x_{17}^2 x_{18}^2 x_{27}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{49}^2 x_{56}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{60}^{(4)} = \left[\frac{i x_{13}^2 x_{26}^2 x_{47}^2 \varepsilon_{123458}}{x_{17}^2 x_{18}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{61}^{(4)} = \left[\frac{i x_{16}^2 x_{24}^2 x_{57}^2 \varepsilon_{123458}}{x_{17}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{48}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{62}^{(4)} = \left[\frac{i x_{14}^4 x_{67}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{57}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{63}^{(4)} = \left[\frac{i x_{24}^2 \varepsilon_{123456}}{x_{16}^2 x_{26}^2 x_{29}^2 x_{37}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{64}^{(4)} = \left[\frac{i x_{13}^4 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{29}^2 x_{36}^2 x_{38}^2 x_{39}^2 x_{48}^2 x_{57}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2} \right]$$

$$\mathcal{I}_{65}^{(4)} = \left[\frac{i x_{13}^2 x_{14}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{19}^2 x_{29}^2 x_{38}^2 x_{39}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{66}^{(4)} = \left[\frac{i x_{14}^2 x_{17}^2 \varepsilon_{123456}}{x_{16}^2 x_{18}^2 x_{19}^2 x_{29}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{67}^{(4)} = \left[\frac{i x_{14}^4 x_{26}^2 \varepsilon_{123456}}{x_{16}^2 x_{17}^2 x_{27}^2 x_{29}^2 x_{38}^2 x_{46}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{68}^{(4)} = \left[\frac{i x_{13}^2 x_{26}^2 \varepsilon_{123457}}{x_{17}^2 x_{18}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{69}^{(4)} = \left[\frac{i x_{16}^2 x_{24}^2 \varepsilon_{123457}}{x_{17}^2 x_{18}^2 x_{26}^2 x_{27}^2 x_{36}^2 x_{46}^2 x_{49}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right]$$

$$\mathcal{I}_{70}^{(4)} = \left[\frac{i x_{13}^2 x_{25}^2 \varepsilon_{123456}}{x_{17}^2 x_{18}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{56}^2 x_{58}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$\mathcal{I}_{71}^{(4)} = \left[\frac{i x_{14}^2 x_{26}^2 x_{37}^2 \varepsilon_{123458}}{x_{17}^2 x_{18}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{46}^2 x_{48}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2} \right]$$

$$(D.0.2)$$

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Note that all these above terms require cyclic addition and we have only provided one term. The corresponding four-loop amplitude graphs are given in Figs.D.2 and D.1.
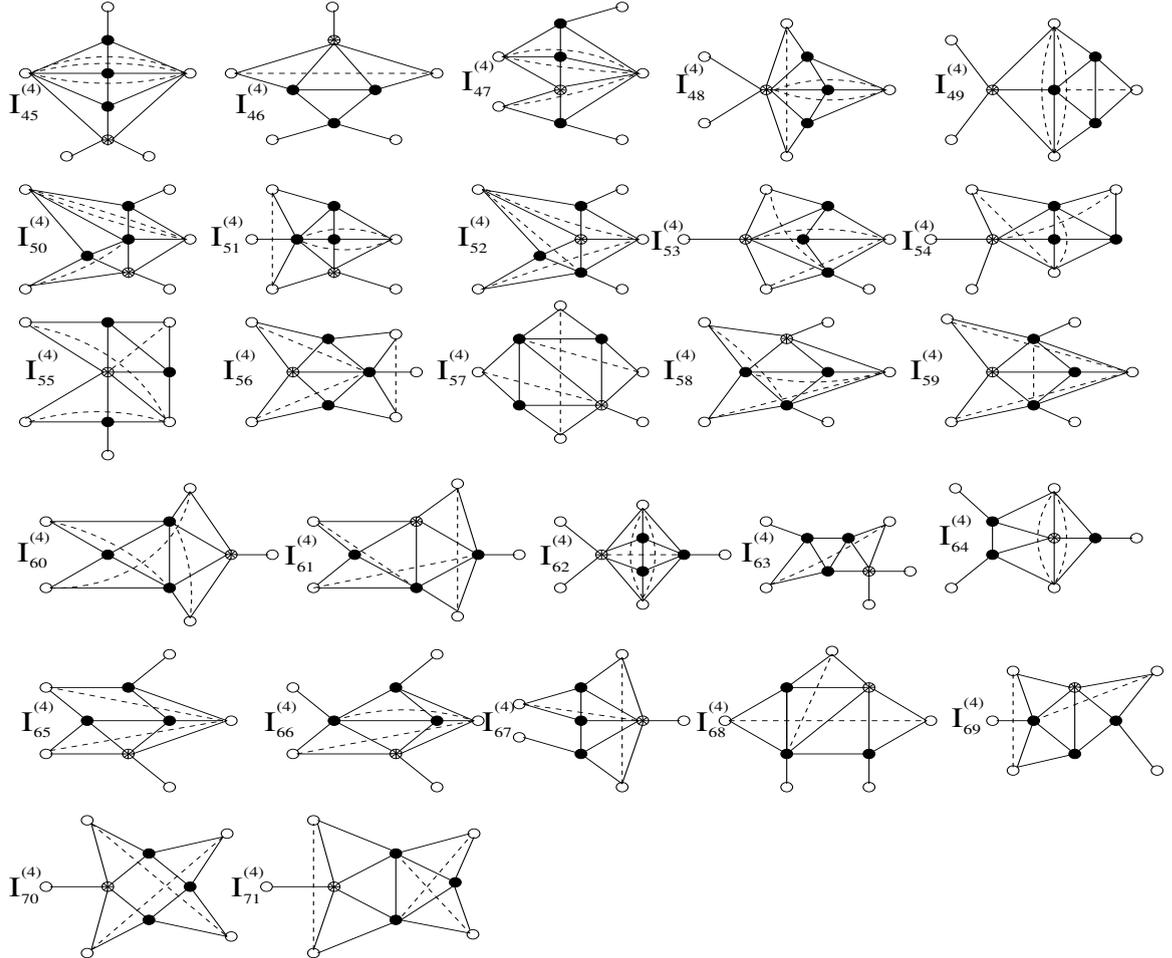
Figure D.1: Four-loop, Five-point, parity-odd amplitude graphs. A starred vertex v indicates a factor $i\varepsilon_{12345v}$
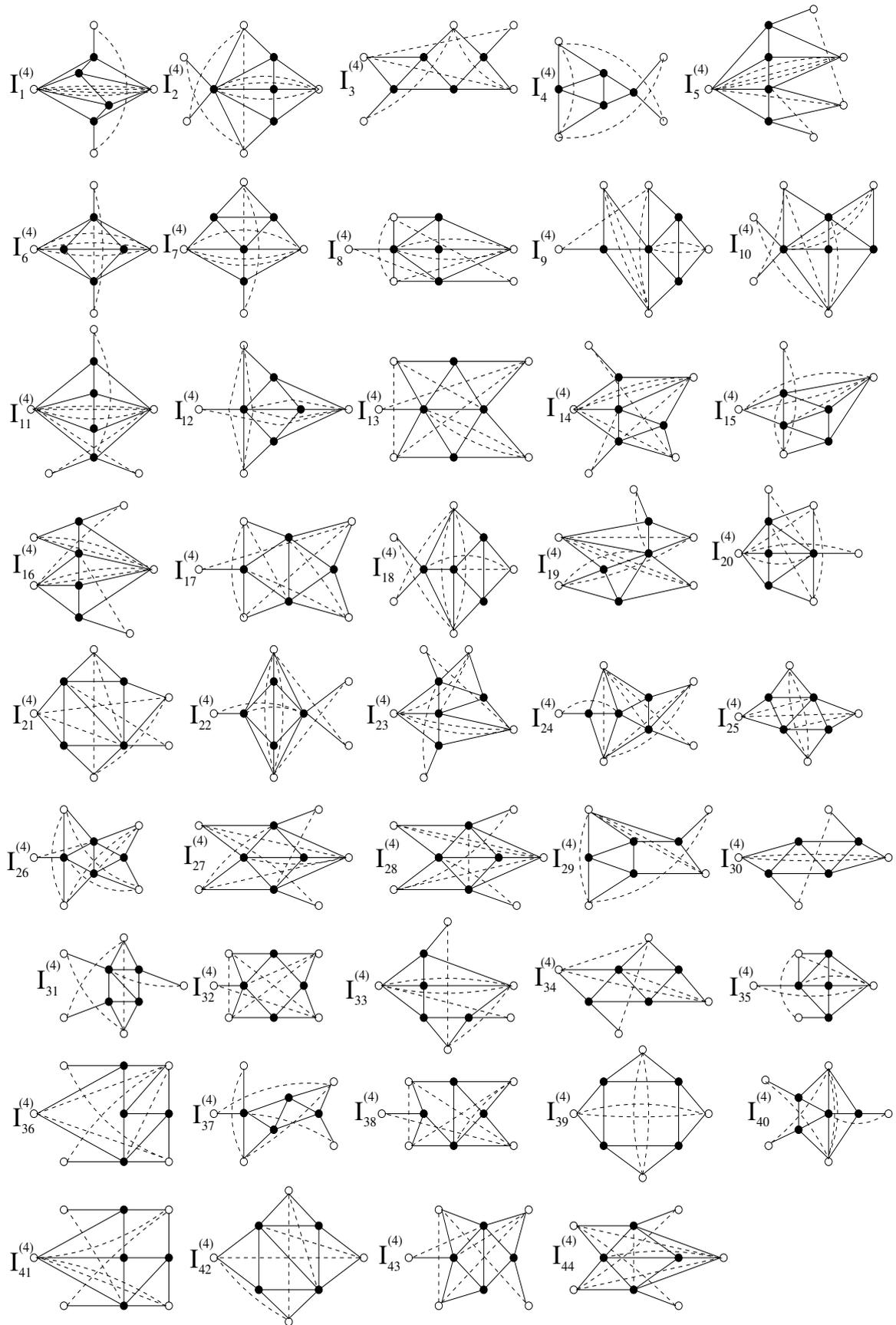
Figure D.2: Four-loop, five-point, parity-even amplitude graphs.

Appendix E

Supercorrelator ↔ Superamplitude Duality

We here explore the correlator/amplitude duality [1, 2, 67, 68, 73, 74] and give more detail and motivation to our equations in Chapter 5. To make the full duality exact we utilise superspace and package the full set of component fields of the energy-tensor multiplet into a single superfield $\mathcal{O}(x, \rho, \overline{\rho}, y) = \text{Tr}(W^2)$ where the trace is over the $\text{SU}(N)$ gauge group, see [93, 94] and references therein. The field strength multiplet $W(x, \rho, \overline{\rho}, y)$ lives on analytic superspace, combining Minkowski space (x) with Grassmann odd coordinates $\rho, \overline{\rho}$ and y coordinates which parametrise the internal symmetry of the N=4 model. Likewise, as we have used in previous chapters, amplitudes connected by supersymmetry can be packaged into a superamplitude expressible in momentum supertwistors [92, 100].

To obtain the full super-duality between any amplitude and the dual correlation function of the above operators, one makes an identification between lightlike coordinate differences on the correlator side with $x_{i,i+1} = p_i$ while sending $\overline{\rho} \to 0$ at all points. The precise identification of the left-handed Grassmann odd coordinates $\{\rho_i\}$ with the odd part of the momentum supertwistors $\{\chi_i\}$ is known [67, 68] but is not needed here, and in the amplitude limits the y coordinates factor out.

We denote the n-point function of energy-momentum multiplets $\mathcal{O}_n$ as $\mathcal{G}_n$ and as such the amplitude/correlator duality states

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{G}_n}{\mathcal{G}_n^{\text{tree}}} = (\mathcal{M}_n)^2 \qquad (\overline{\rho} = 0) \qquad \text{(E.0.1)}$$

On the correlator side of the equation $\mathcal{G}_n$ denotes a superspace object containing, n-point correlators of any operators in the energy-momentum multiplet in a single object, where some components go to zero once $\bar{\rho} \to 0$. The amplitude side ($\mathcal{M}_n$) contains all n-point amplitudes in the theory packaged in one superspace object - the superamplitude. To be precise $\mathcal{M}_n$ is the full superamplitude divided by the tree-level amplitude, so the leading term of $\mathcal{M}_n$ is 1. Both sides of this equation have expansions both in powers of the odd superspace variables and the coupling constants, firstly expanding in odd superspace variables

$$\mathcal{G}_n = \sum_{k=0}^{n-4} G_{n;k} \qquad \mathcal{M}_n = \sum_{k=0}^{n-4} M_{n;k} \qquad \text{(E.0.2)}$$

Where $G_{n;k}$ and $M_{n;k}$ contain $4k$ powers of the odd superspace variable, in particular $M_{n;k}$ is the N^{k}MHV superamplitude.

By differentiation in the coupling constant it can be shown that

$$G_{n;k}^{(\ell)} = \frac{a^\ell}{\ell!} \prod_{i=1}^{\ell} \left(\int d^4 x_{n+i} d^4 \rho_{n+i} \right) G_{n+\ell,k+\ell}^{(0)} \qquad \ell > 0 \qquad \text{(E.0.3)}$$

where the superscript (ℓ) indicates the loop order. To be explicit: the ℓ-loop correction to an energy-momentum, n-point function is given by a superspace integral over a Born level correlator of the same type, simply with more points. As such we may consider various n-gon limits of the same correlator. We currently know very little about the correlation functions $G_{n;k}$ with $k < n - 4$. However, following [69, 70] we have a wealth of information about the "maximally nilpotent" case $k = n - 4$. We will use this mechanism to construct 4-point and 5-point amplitudes from the correlators $G_{n;n-4}^{(0)}$ which were originally elaborated on for the higher-loop integrands of the four-point function [69]

$$G_{4;0}^{(\ell)} = \frac{a^\ell}{\ell!} \prod_{i=1}^{\ell} \left(\int d^4 x_{4+i} d^4 \rho_{4+i} \right) G_{4+\ell;\ell}^{(0)}$$
$$G_{5;1}^{(\ell-1)} = \frac{a^{(\ell-1)}}{(\ell-1)!} \prod_{i=1}^{\ell-1} \left(\int d^4 x_{5+i} d^4 \rho_{5+i} \right) G_{4+\ell;\ell}^{(0)} \qquad \text{(E.0.4)}$$

According to [32, 69–72, 75, 76, 86] the Born level correlator with maximum $k = n - 4$ (maximally nilpotent piece) has the form

$$G_{4+\ell;\ell|\rho_5^4 \cdots \rho_{4+\ell}^4}^{(0)} = \mathcal{I}_{1234} \rho_5^4 \ldots \rho_{4+\ell}^4 f^{(\ell)}(x_1, \ldots, x_{4+\ell}) \qquad \text{(E.0.5)}$$

where

$$\mathcal{I}_{1234} = \frac{2(N^2 - 1)}{(4\pi^2)^4}(x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2)\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{14}^2}{x_{14}^2}x_{13}^2 x_{24}^2 + \ldots\right) \quad \text{(E.0.6)}$$

Here the dots indicate terms subleading in both the 4-gon limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \to 0$ and the 5-gon limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{45}^2, x_{51}^2 \to 0$ which we are concerned with.

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